4.3 Linear Combinations and Linear Independence

vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n$

the linear combination of them is $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \cdots + c_n\vec{v}_n$

if they are linearly independent, then $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$

implies $c_1 = c_2 = \cdots = c_n = 0$ as the only possibility.

> Linear independence → linear combo $= \vec{0}$
> if and only if $c_1 = c_2 = \cdots = c_n = 0$

if the $\vec{v}_i$ are linearly independent, then if they are columns of a matrix, there will be exactly as many pivots as there are vectors

→ ~~other wis~~ otherwise, the solution to $[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

will NOT be unique

for example, $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 \\ 0 & \boxed{-2} \end{bmatrix}$$

two pivots
two vectors
so $\vec{v}_1, \vec{v}_2$ are linearly independant

because $c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$

becomes $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

unique solution
$c_1 = c_2 = 0$
(trivial solution)

another example : $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\vec{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

are they linearly independent ?

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 2 \end{bmatrix}$$

3 pivots, 4 vectors, NOT linearly independent

(BUT, a subset of those 4 <u>may</u> still be linearly independent)

here, vectors have 3 components, so matrix has 3 rows

so max number of pivots is 3 because we don't have enough rows

> So, if vectors m components, and there are more than m vectors, then the vectors are always linearly dependent

if there are fewer than m vectors, or m vectors then they may or may not be linearly independent

for example, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ two components two vectors

NO, they are multiples of each other

so, for example, $2\vec{v}_1 - \vec{v}_2 = \vec{0}$ $\rightarrow$ $c_1 = c_2 = 0$ is NOT only solution since number of pivots is the key, that means the determinant can also be used to determine linearly independance

for example, $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has determinant 1

(two pivots so nonzero determinant)

$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ determinant is $(1)(2) - (1)(2)$
$= 0$

or reduce first $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ det $= 0$

$\boxed{\text{determinant} = 0 \rightarrow \text{vectors who are columns of matrix are NOT linearly independent}}$

square matrix only

example: $\vec{v}_1 = \begin{bmatrix} -1 \\ -17 \\ -3 \\ 9 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 14 \\ 7 \\ 2 \\ -2 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 15 \\ 5 \\ 1 \\ -2 \end{bmatrix}$

linearly independent?

3 4-component vectors ⟶ may or may not be independent

$$\begin{bmatrix} -1 & 14 & 15 \\ -17 & 7 & 5 \\ -3 & 2 & 1 \\ 9 & -2 & -2 \end{bmatrix}$$

not square, so can't find determinant

reduce to count pivots

$$\longrightarrow \cdots \longrightarrow \begin{bmatrix} \boxed{-1} & 14 & 15 \\ 0 & \boxed{-231} & -250 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

3 pivots

3 vectors

so linearly independent

??? free variable(s)?

$$c_1 \vec{v_1} + c_2 \vec{v_2} + c_3 \vec{v_3} = \vec{0}$$

$$\begin{bmatrix} -1 & 14 & 15 & 0 \\ -17 & 7 & 5 & 0 \\ -3 & 2 & 1 & 0 \\ 9 & -2 & -2 & 0 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} -1 & 14 & 15 & 0 \\ 0 & -231 & -250 & 0 \\ 0 & 0 & 1 & 0 \\ \boxed{0 \quad 0 \quad 0 \quad 0} \end{bmatrix}$$

zero row : variable(s) <u>without</u> pivots in their columns are free

here, there are pivots in 1st, 2nd, 3rd columns

so, c_1, c_2, c_3 are <u>NOT</u> free

in fact, ignore last row, we see $c_1 = c_2 = c_3 = 0$

<u>Span</u> is another very important concept

we say the vectors $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

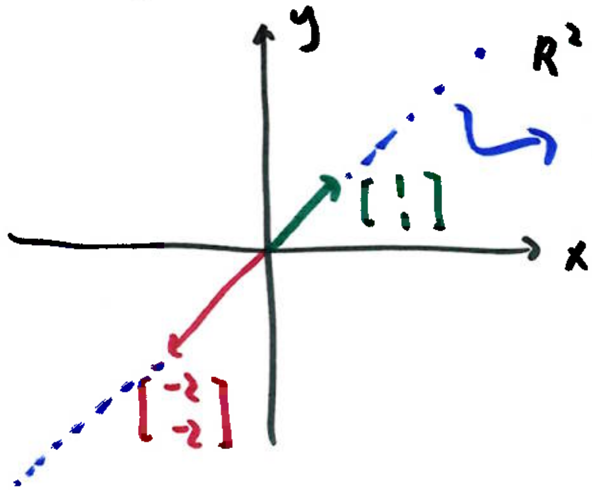
span R^2 because we can make every possible R^2 vector with linear combos of $\vec{i}$ and $\vec{j}$

→ we can <u>fill up</u> R^2 using $\vec{i}$ and $\vec{j}$

so we write $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = R^2$

likewise, $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = R^3$

does $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right\}$ span R^2 ?



linear combos of them

we cannot make any R^2 vector outside that line

so, NO, they do NOT span R^2

what they do span is a subspace of R^2 that is
the blue dotted line on previous page

does $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ span R^2 ?

yes, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ span R^2, adding extra vector
does NOT change that fact

likewise $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\}$
also spans R^2