4.4 Bases and Dimension of a Vector Space

span : we have vectors to fill up a vector space

for example, $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span R^2

because every vector in R^2 is a linear combo of $\vec{i}$ and $\vec{j}$

$\begin{bmatrix} 5 \\ 7 \end{bmatrix} = 5\vec{i} + 7\vec{j}$

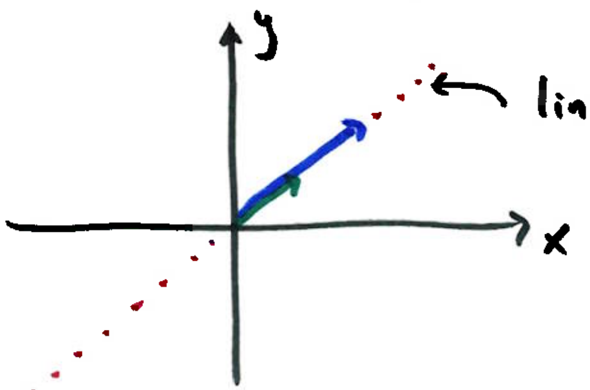
R^2

minimum two vectors but may not be enough

R^2 is also spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (more than needed)

but R^2 is <u>NOT</u> spanned by just $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (can't cover y-axis)

R^2 is <u>NOT</u> spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$



linear combos of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Similarly, R^3 is spanned by $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

R^3 is also spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

R^3 ← minimum three

but NOT spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (missing z)

and NOT spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ (missing z)

So, we see than for R^n, we need <u>at least</u> n vectors

and <u>exactly</u> n if they are <u>linearly independent</u>

and <u>more than n</u> if they are not linearly independent

the n linearly independent vectors are called <u>basis vectors</u> or <u>bases</u>

$\vec{i}, \vec{j}, \vec{k}$ are the <u>standard basis vectors</u> for R^n

but given R^n, the basis vectors are NOT unique

for example, any 3 linearly independent vectors span R^3

and can serve as bases for R^3

R^2 example: $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are one set of basis vectors

so are $\vec{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ (two linearly independent vectors)

we can express any R^2 using any basis

$$\begin{bmatrix} 10 \\ -7 \end{bmatrix} = 10\vec{i} - 7\vec{j} = \frac{57}{10}\vec{u} - \frac{7}{5}\vec{v}$$

vectors in R^n are NOT uniquely expressed, but are unique within a chosen basis vector set

the <u>dimension</u> of a vector space is the <u>number of vectors in the basis set</u>

R^2: needs two basis vectors $\rightarrow$ two-dimensional vector space

R^3: " three " " $\rightarrow$ three-dimensional " "

in R^n, the word <u>dimension</u> has the same meaning as our everyday language but not always the case

<u>example</u> Find the dimension and ~~&~~ a basis for the subspace of R^3 given by the plane $3x+2y+z=0$

$3x+2y+z=0$ is a subspace that contains vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $3x+2y+z=0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -3x-2y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} x \\ -\frac{3}{2}x-\frac{1}{2}z \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ -\frac{3}{2} \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{3}y-\frac{1}{3}z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

so, basis of this subspace are

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 1 \\ -3/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix} \right\}$$

$$\text{or } \left\{ \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

or any scalar multiple of the above

so, the dimension of this subspace is <u>two</u> (two vectors in basis)

<u>example</u> Solution space of

$$\begin{aligned} x_1 - 2x_2 - 5x_3 &= 0 \\ 2x_1 - 3x_2 - 13x_3 &= 0 \end{aligned}$$

solution space : contains ALL solutions $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

solve : $\begin{bmatrix} 1 & -2 & -5 & 0 \\ 2 & -3 & -13 & 0 \end{bmatrix}$

$$\longrightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \begin{bmatrix} \boxed{1} & -2 & -5 & 0 \\ 0 & \boxed{1} & -3 & 0 \end{bmatrix} \end{array}$$

column 3 has no pivot, so let $x_3 = r$ (free)

row 2: $x_2 - 3x_3 = 0$

$x_2 = 3x_3 = 3r$

row 1: $x_1 - 2x_2 - 5x_3 = 0$

$x_1 = 2x_2 + 5x_3$

$= 6r + 5r = 11r$

so, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11r \\ 3r \\ r \end{bmatrix} = r \begin{bmatrix} 11 \\ 3 \\ 1 \end{bmatrix}$

solution space has basis $\begin{bmatrix} 11 \\ 3 \\ 1 \end{bmatrix}$ and is one-dimensional

choosing x_1 or x_2 to be free will give us alternative bases linear independance : sum to $\vec{0}$ only if all coefficients are zero

span : enough vectors to fill up or cover a vector space
(may be more than needed)

basis : exactly enough (no more and no less than needed)
to cover the vector space (must be linearly indp)

dimension : # of vectors in the basis set
(NOT always # of components of the vectors)