# 4.5 Row and Column Spaces

given matrix A, the vector space spanned by its rows is called the row space of A : Row(A)

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

rows: $[1 \;\; 2]$, $[0 \;\; 1]$

since they are linearly independent
we know they span $R^2$
and they are ~~the~~ a basis of Row(A)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Row(A) is spanned by $[1 \;\; 0]$, $[0 \;\; 0]$

So Row(A) is $R^1$ and basis is $[1 \;\; 0]$

this one is not as easy to tell what the row space is

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 4 & 3 \\ 2 & 2 & 5 \end{bmatrix}$$

Row(A) ?
basis of Row(A) ?

$\text{Row}(A) = \text{span}\{[1\ -2\ \ 2],\ [1\ \ 4\ \ 3],\ [2\ \ 2\ \ 5]\}$ $R^3$ ?

a basis? → find linearly indp vectors in Row(A)

Gaussian elimination / row operations uncovers linear independance

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 4 & 3 \\ 2 & 2 & 5 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 6 & 1 \\ 0 & 0 & 0 \end{bmatrix} = B$$

note: $A \neq B$ but $A$ and $B$ are <u>row equivalent</u>

row equivalent → same row space because row operations are linear combos of row vectors which do <u>NOT</u> affect the row space

$\text{Row}(B) = \text{span}\{[1\ -2\ \ 2],\ [0\ \ 6\ \ 1]\}$

pivot rows → linearly independant

basis for $\text{Row}(B) = \text{Row}(A)$ is $\{[1\ -2\ \ 2],\ [0\ \ 6\ \ 1]\}$

To find basis for Row(A), perform row ops to get to echelon matrix, then read the pivot rows the rank of Row(A) is the number of vectors in a basis for Row(A) $\rightarrow$ number of pivots

Likewise, the vector space spanned by columns of A is called the column space $\rightarrow$ Col(A)

Solving $A\vec{x} = \vec{b}$ is equivalent to checking if $\vec{b}$ is in Col(A) and find a particular linear combo of columns.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 8 & 8 \end{bmatrix} \qquad \mathrm{Col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} \right\}$$

basis ?

row operations, in general, do NOT preserve column space (but they do preserve row space)

So, we cannot just reduce and read pivot columns

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 8 & 8 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = B$$

notice col(B) does NOT ~~reach~~ have z-component
whereas Col(A) does
so, after row ops, column space changed

however, row operations do NOT affect the linear independence of the columns

so, here, pivot columns are columns 1 and 2, so columns 1 and 2 are linearly independent for BOTH A and B

so, columns 1 and 2 of A are linearly independent and they form a basis for Col (A)

To find basis for Col(A), perform row operations and identify pivot columns in the echelon form, then the corresponding columns in the original matrix form a basis for Col (A)

**example**

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 5 & -9 & 10 \\ 1 & 6 & -29 & -11 \end{bmatrix}$$

find a basis for Row(A) and Col(A)

$$\xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -12 & 7 \\ 0 & 0 & 0 & -59 \end{bmatrix}$$

pivot rows: all three

so, basis for Row(A) are the pivot rows of echelon form: $\{[1\ 1\ 1\ 1], [0\ 2\ -12\ 7], [0\ 0\ 0\ -59]\}$

pivot columns: 1, 2, 4

so, basis for ~~Ro~~ Col(A) are the corresponding columns of A: $\underbrace{\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ -11 \end{bmatrix} \right\}}$

same as identifying which of the vectors that form columns of A are linearly indp example

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

which of these are linearly independent?

$\longrightarrow$ find a basis for Col(A) where

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 4 & 1 \\ 2 & 1 & 3 & 2 \\ 2 & 2 & 2 & 1 \\ 2 & 1 & 3 & 4 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 3 & 2 & 4 & 1 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccc} \uparrow & \uparrow & & \uparrow \end{array}$$

pivot cols: 1, 2, 4

so cols 1, 2, 4 $(\vec{v}_1, \vec{v}_2, \vec{v}_4)$ are linearly independent alternative way to find basis for Col(A)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -9 & 17 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & -9 \\ 3 & 17 \end{bmatrix}$$

finding basis for Row($A^T$) is equivalent to

finding basis for Col(A) and vice versa

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & -9 \\ 3 & 17 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 \\ 0 & -17 \\ 0 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 \\ 0 & -17 \\ 0 & 0 \end{bmatrix}$$

pivot rows : 1, 2

so basis for Row($A^T$)

= Col(A) is

$$\left\{ \begin{bmatrix} 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & -17 \end{bmatrix} \right\}$$

as rows

or $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -17 \end{bmatrix} \right\}$ as columns