5.1 Intro to Linear Second Order Equations

2nd-order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x)$$

cannot contain y

we first focus on the associated homogeneous equation

right side is zero

$$\boxed{y'' + p(x)y' + q(x)y = 0}$$

this equation has two linearly independent solutions $\boxed{y_1 \text{ and } y_2}$

the solution y_1 and y_2 span (and are basis) of the solution space

(the solution space is a vector space → "vectors" are functions)

linear combos of y_1 and y_2 for the general solution

$$\boxed{y = c_1 y_1 + c_2 y_2}$$

<u>example</u> $y'' - y = 0$

verify the solutions are $y_1 = e^x$ and $y_2 = e^{-x}$

just like w/ 1st-order, solutions must satisfy the differential equation

$y_1 = e^x \quad y_1' = e^x \quad y_1'' = e^x$ sub into $y'' - y = 0$

$e^x - e^x = 0$ true, so $y_1 = e^x$ is a solution

$y_2 = e^{-x} \quad y_2' = -e^{-x} \quad y_2'' = e^{-x}$ sub into $y'' - y = 0$

$e^{-x} - e^{-x} = 0$ true, so $y_2 = e^{-x}$ is a solution

they span the solution space that contains all general solutions $y = c_1 y_1 + c_2 y_2 = c_1 e^x + c_2 e^{-x}$ c_1, c_2 are constants

c_1 and c_2 depend on boundary or <u>initial conditions</u>

$$\left.\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_0' \end{aligned}\right\}$$ need Two

for example, for this differential equation

if $y(0)=1$ and $y'(0)=0$

$$y = c_1 e^x + c_2 e^{-x} \qquad \underbrace{y' = c_1 e^x - c_2 e^{-x}}$$

$y(0)=1 \rightarrow 1 = c_1 + c_2$

$y'(0)=0 \rightarrow 0 = c_1 - c_2$

solve the system $\begin{array}{l} c_1 + c_2 = 1 \\ c_1 - c_2 = 0 \end{array}$ for c_1, c_2

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix}$$

$-2c_2 = -1$

$c_2 = \frac{1}{2}$

$c_1 + c_2 = 1$

$c_1 = 1 - c_2 = \frac{1}{2}$

so, general solution is $y = c_1 e^x + c_2 e^{-x} = \frac{1}{2} e^x + \frac{1}{2} e^{-x}$

linearly indp: for functions → not multiples of each other

$y_1 = e^x$, $y_2 = e^{-x}$ clearly not multiples of each other

same w/ $y_1 = \cos x$ $y_2 = \sin x$

more generally, the way to check is to evaluate the <u>Wronskian</u>

functions f, g

$$W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \neq 0$$ for all values of x on some interval then f and g are linearly indp

$f = e^x$ $g = e^{-x}$

$f' = e^x$ $g' = -e^{-x}$

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2 \neq 0 \text{ for } x$$

so e^x and e^{-x} are linearly indp on $(-\infty, \infty)$

Simplest case of 2nd order: linear constant-coefficient homogeneous

$$y'' + ay' + by = 0$$

y: function relating to its first and second derivs → exponential

$y = e^{rx}$ is solution form

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$r^2 e^{rx} + are^{rx} + be^{rx} = 0$

$e^{rx}(r^2 + ar + b) = 0$

$e^{rx} \neq 0$ for any r or x

so, $\boxed{r^2 + ar + b = 0 \rightarrow \text{characteristic equation}}$

solutions are $\boxed{y_1 = e^{r_1 x}, \; y_2 = e^{r_2 x}}$

where r_1, r_2 are solutions to characteristic eq.

<u>example</u> $y'' + 3y' + 2y = 0$

characteristic eq: $r^2 + 3r + 2 = 0$ order of deriv $\rightarrow$ power of r

$(r+2)(r+1) = 0$

$r_1 = -2 \qquad r_2 = -1$

solutions: $y_1 = e^{r_1 x} = e^{-2x}$

$y_2 = e^{r_2 x} = e^{-x}$

general solutions: $\boxed{y = c_1 e^{-2x} + c_2 e^{-x}}$

this is the case when r's are <u>real and distinct</u>

next, the repeated roots case

<u>example</u> $4y'' + 4y' + y = 0$

characteristic eq : $4r^2 + 4r + 1 = 0$

(if you forget, just plug $y = e^{rx}$ into equation, the characteristic eq. will result)

$(2r + 1)(2r + 1) = 0$

$r_1 = -\frac{1}{2}$ $r_2 = -\frac{1}{2}$ repeated

form y_1 as usual : $y_1 = e^{r_1 x} = e^{-\frac{1}{2}x}$

if we form y_2 as usual, y_1 and y_2 are <u>NOT</u> indp

form y_2 : $y_2 = x e^{r_2 x} = x e^{-\frac{1}{2}x}$

general solution: $y = c_1 e^{-\frac{1}{2}x} + c_2 \underline{\underline{x}} e^{-\frac{1}{2}x}$