5.2 General Solutions of Linear Equations

n^{th}-order linear : $y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \cdots + p_{n-1}(x)y' + p_n(x)y = F(x)$

cannot contain y

for example, $y''' + x^2y'' + e^xy' + 3y = \cos(x)$

the associated homogeneous equation : right side is zero

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$$

the solution of the nonhomogeneous equation is the sum of
the solution of the associated homogeneous eq. and another
solution due to the right side.

the n^{th}-order homogeneous eq. has n linearly independent solutions

$$y_1, y_2, y_3, \ldots, y_n$$

the linear combination of them → general solution $y = C_1y_1 + C_2y_2 + \cdots + C_ny_n$

this is called the <u>Principle of Superposition</u>

functions $f_1, f_2, f_3, \ldots, f_n$ are linearly independent on an interval if the only way to sum to zero is the <u>trivial combination</u>

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0 \longrightarrow c_1 = c_2 = \cdots = c_n = 0 \text{ for}$$

ALL x on the interval

for example, $f_1 = 1$, $f_2 = x$, $f_3 = x^2$ are linearly independent on $(-\infty, \infty)$ because the only way to sum to zero for ALL x on $(-\infty, \infty)$ is if $c_1 = c_2 = c_3 = 0$

$$c_1(1) + c_2(x) + c_3(x^2) = 0$$

($c_1 = 0$, $c_2 = 1$, $c_3 = 1$ at $x = -1$ is possible but that set of c's don't work for other x's on $(-\infty, \infty)$)

another example: $f_1 = 1, \; f_2 = \cos^2 x, \; f_3 = \sin^2 x$

to form zero w/ $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$

$$c_1 + c_2 \cos^2 x + c_3 \sin^2 x = 0$$

we know $\cos^2 x + \sin^2 x = 1$ for ALL x

than if $c_1 = -1, \; c_2 = 1, \; c_3 = 1$

$$-1 + \cos^2 x + \sin^2 x = 0 \rightarrow \text{true for all } x$$

so, $c_1 = -1, \; c_2 = 1, \; c_3 = 1$ work for all x

therefore, since they are not all zeros, we have found a nontrivial combo that works for all x

so $1, \; \cos^2 x, \; \sin^2 x$ are <u>NOT</u> linearly independent

for higher number of functions, the Wronskian works better functions $f_1, f_2, f_3, \dots, f_n$ (n of them)

$$W = \begin{vmatrix} f_1 & f_2 & & f_n \\ f_1' & f_2' & & f_n' \\ f_1'' & f_2'' & \cdots & f_n'' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & & f_n^{(n-1)} \end{vmatrix}$$

if $W \neq 0$ for an interval, then the functions are linearly independent

if $W = 0$ for all x on an interval, then the functions are linearly <u>dependent</u> on that interval

for example, $f_1 = 1$, $f_2 = x$, $f_3 = x^2$

$$W = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = (1)(1)(2) = 2 \neq 0 \text{ for ANY } x$$

so f_1, f_2, f_3 are independent for all x's $\rightarrow$ on $(-\infty, \infty)$

$f_1 = 1, \quad f_2 = \cos^2 x, \quad f_3 = \sin^2 x$

$$W = \begin{vmatrix} 1 & \cos^2 x & \sin^2 x \\ 0 & -2\cos x \sin x & 2\sin x \cos x \\ 0 & 2\sin^2 x - 2\cos^2 x & -(2\sin^2 x - 2\cos^2 x) \end{vmatrix} = \begin{vmatrix} -2\cos x \sin x & 2\sin x \cos x \\ 2\sin^2 x - 2\cos^2 x & -(2\sin^2 x - 2\cos^2 x) \end{vmatrix}$$

$= 0$ for all x $\rightarrow$ dependent for ALL x

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general solution: $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$

to find c's, we need n initial conditions

for example, $y^{(3)} + y' = 0$ has solutions $y_1 = 1$, $y_2 = \cos x$, $y_3 = \sin x$

and initial conditions $y(0) = 1$, $y'(0) = -1$, $y''(0) = 0$

general solution: $y = c_1 \cdot 1 + c_2 \cdot \cos x + c_3 \cdot \sin x$

$y = C_1 + C_2 \cos x + C_3 \sin x$ need y', y'' to apply initial conditions

$y' = -C_2 \sin x + C_3 \cos x$

$y'' = -C_2 \cos x - C_3 \sin x$

$y(0) = 1 \longrightarrow 1 = C_1 + C_2 \longrightarrow C_1 = 1$

$y'(0) = -1 \longrightarrow -1 = C_3 \rightarrow C_3 = -1$

$y''(0) = 0 \longrightarrow 0 = -C_2 \longrightarrow C_2 = 0$

$\boxed{y = 1 - \sin x}$ this is called a <u>particular solution</u>

(solved C's from initial conditions)

Reduction of Order

given $y'' + p(x)y' + q(x)y = 0$

if we know (somehow) one solution $\rightarrow y_1$

the second solutions can be found by $y_2 = v(x)\, y_1$

finding $v(x) \rightarrow$ finding y_2

example $x^2y'' + xy' - 9y = 0$ (Euler's equation $\rightarrow$ non constant coeffecients)

$y_1 = x^3$ find y_2

assume $y_2 = v\,y_1 = v\,x^3$ plug into the equation

$y_2' = 3vx^2 + v'x^3$

$y_2'' = 6vx + 6v'x^2 + v''x^3$

$$x^2(6vx + 6v'x^2 + v''x^3) + x(3vx^2 + v'x^3) - 9(vx^3) = 0$$

$$\cancel{6vx^3} + 6v'x^4 + v''x^5 + \cancel{3vx^3} + v'x^4 - \cancel{9vx^3} = 0$$

$x^5 v'' + 7v'x^4 = 0$

rewrite: $v'' = -7v'x^{-1}$

$$\frac{d(v')}{dx} = \frac{-7(v')}{x}$$ separable in v' and x

$$\frac{1}{v'}\, d(v') = -\frac{7}{x}\, dx$$

$\vdots$

$v' = c x^{-7}$ $y_2 = v y_1$ need v

$$v = \int c x^{-7}\, dx$$

$$v = -\frac{c}{6} x^{-6} + D$$

choose ANY c, D that is convenient (except those that lead to $v = 0$)

choose $c = -6$

$D = 0$

so $v = x^{-6}$

so, $y_2 = v y_1 = x^{-6} x^3 = x^{-3}$