# 5.3 Homogeneous Equations with Constant Coefficients

as long as the coefficients of the linear equation are constant,
we can always assume solutions of the form $y = e^{rx}$

$$\left.\begin{aligned} &y'' + 5y' - 2y = 0 \\ &y''' + 10y'' - 5y' + 17 = 0 \\ &y^{(4)} - 100y = 0 \end{aligned}\right\} \; y = e^{rx} \text{ are the solutions}$$

2nd order: $ay'' + by' + cy = 0 \rightarrow$ Characteristic eq $\underbrace{ar^2 + br + c = 0}$

two roots: $r_1, r_2$
(real and distinct, repeated, or complex)

3rd order: $ay''' + by'' + cy' + dy = 0 \rightarrow \underbrace{ar^3 + br^2 + cr + d = 0}$

three roots: $r_1, r_2, r_3$
(repeated real, distinct real, or some complex)

$n^{th}$ order: $n$ roots solutions are $y_1 = e^{r_1 x}, y_2 = e^{r_2 x}, \ldots, y_n = e^{r_n x}$

<u>example</u> $2y'' - 3y' = 0$

$2r^2 - 3r = 0 \longrightarrow r(2r-3) = 0 \quad r_1 = 0, \; r_2 = 3/2$ (distinct)

solutions: $y_1 = e^{r_1 x}$ so $y_1 = 1$

$y_2 = e^{r_2 x}$ so $y_2 = e^{3/2\, x}$

general solution: $y = c_1 + c_2 e^{3/2\, x}$

<u>example</u> $4y'' - 12y' + 9y = 0$

$4r^2 - 12r + 9 = 0$

$(2r-3)(2r-3) = 0 \quad r_1 = 3/2, \; r_2 = 3/2$ (repeated)

$y_1 = e^{r_1 x} = e^{3/2\, x}$

$y_2 = xe^{r_2 x} = xe^{r_1 x} = xe^{3/2\, x}$

gen. solution $y = c_1 e^{3/2\, x} + c_2 x e^{3/2\, x}$

**example** $y^{(4)} - 8y''' + 16y'' = 0$

characteristic eq. $r^4 - 8r^3 + 16r^2 = 0$

$$r^2(r^2 - 8r + 16) = 0$$

$$(r)(r)(r-4)(r-4) = 0$$

$$r_1 = 0,\ r_2 = 0,\ r_3 = 4,\ r_4 = 4$$

$y_1 = e^{r_1 x} = 1$

$y_2 = e^{r_2 x}$ but $r_1 = r_2$, so $y_2 = x e^{r_2 x} = x$

each repeat multiply by $x$

$y_3 = e^{r_3 x} = e^{4x}$

$y_4 = x e^{4x}$

(if $r=0$ had shown up 3 times, $y_1 = 1$, $y_2 = x$, $y_3 = x^2$)

general solution: $y = C_1 + C_2 x + C_3 e^{4x} + C_4 x e^{4x}$

__example__ If the general solution of a linear constant coefficient equation is $y = (c_1 + c_2 x + c_3 x^2) e^{-2x}$

what are the roots of the characteristic eq?

what is the differential equation?

3 solutions $\longrightarrow$ 3rd order

$$\left.\begin{aligned} y_1 &= e^{-2x} \longrightarrow r_1 = -2 \\ y_2 &= x e^{-2x} \longrightarrow r_2 = -2 \\ y_3 &= x^2 e^{-2x} \longrightarrow r_3 = -2 \end{aligned}\right\}$$

characteric eq: $(r+2)^3 = 0$

$$(r+2)(r^2+4r+4) = 0$$

$$r^3 + 4r^2 + 4r + 2r^2 + 8r + 8 = 0$$

$$r^3 + 6r^2 + 12r + 8 = 0$$

$$y''' + 6y'' + 12y' + 8y = 0$$

<u>example</u> $y''' + 5y'' - 100y' - 500y = 0$

$1r^3 + 5r^2 - 100r - 500 = 0$ cubic is not always easy to factor

lucky situation: ratio between 1st, 2nd and 3rd, 4th is the same

(2nd is 5 times 1st, 3rd is 5 times 4th)

factor $r^2$ from first two

factor 3rd coeff from last two

$r^2(r+5) - 100(r+5) = 0$

$(r+5)(r^2-100) = 0$

$r_1 = -5$, $r_2 = 10$, $r_3 = -10$

$y = c_1 e^{-5x} + c_2 e^{10x} + c_3 e^{-10x}$

if the coefficients are not patterned like the above, usually we need to guess and check and then do long division example $3y''' + 4y'' - 5y' - 2y = 0$

$3r^3 + 4r^2 - 5r - 2 = 0$

find a small integer solution (e.g. $r = \pm 1$, $r = \pm 2$) and check

so, for example, does $r = 1$ work?

yes, $3 + 4 - 5 - 2 = 0$ which is true

$(r-1)(ar^2 + br + c) = 0$

$(r-1)(ar^2 + br + c) = 3r^3 + 4r^2 - 5r - 2$

$a = ?$
$b = ?$
$c = ?$

$$ar^2 + br + c = \frac{3r^3 + 4r^2 - 5r - 2}{r - 1}$$

polynomial long divison

$$
\begin{array}{r}
3r^2 + 7r + 2 \\
r - 1 \overline{\big)\, 3r^3 + 4r^2 - 5r - 2} \\
-(3r^3 - 3r^2) \\
\hline
7r^2 - 5r - 2 \\
-(7r^2 - 7r) \\
\hline
2r - 2 \\
-(2r - 2) \\
\hline
0
\end{array}
$$

so, $$\frac{3r^3+4r^2-5r-2}{r-1} = 3r^2+7r+2$$

$$3r^3+4r^2-5r-2 = (r-1)(3r^2+7r+2) = 0$$

$$(r-1)(3r+1)(r+2) = 0$$

$$r_1 = 1, \quad r_2 = -1/3, \quad r_3 = -2$$

$$y = c_1 e^{x} + c_2 e^{-1/3 x} + c_3 e^{-2x}$$

next time: complex roots, e.g. $r = \pm i$, $y = e^{ix}$ ?