# 5.3 Homogeneous Eqs. with Constant Coefficients (part 2)

$n^{th}$ order $\rightarrow$ $n$ solutions of the form $y = e^{rx}$

where $r$ is a root of the characteristic eq.

(auxiliary eq.)

so far, we have seen: distinct real roots

repeated real roots

complex roots? $\quad y'' + 100y = 0$

$r^2 + 100 = 0 \qquad r^2 = -100 \qquad r = 10i,\ -10i$

where $i^2 = -1$

Solutions are formed the same way: $e^{rx}$

if $\quad r = a + bi \qquad$ ($a$ is the real part and $b$ is the imaginary part)

$$y = e^{rx} = e^{(a+bi)x} = e^{ax}\underbrace{e^{i(bx)}}_{?} \rightarrow \text{real but looks complex}$$

$$e^{ix} = ?$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \quad (\text{Taylor series for } e^{x})$$

$$\left.\begin{aligned} i &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \\ i^5 &= i \\ i^6 &= -1 \end{aligned}\right\} \text{repeating}$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots$$

$$= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right)}_{\cos x} + i\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right)}_{\sin x}$$

$$\boxed{e^{ix} = \cos x + i \sin x}$$ Euler's formula

$$e^{i(bx)} = \cos bx + i \sin bx$$

$$\boxed{e^{(a+bi)x} = e^{ax} e^{i(bx)} = e^{ax}(\cos bx + i \sin bx)}$$

if $r = a+bi$ is a root of the characteristic eq, then

so is $r = a-bi$ complex roots always appear in <u>conjugate pairs</u>

<u>example</u> $y'' + 100y = 0$

$r^2 + 100 = 0 \quad r^2 = -100 \quad r_1 = 10i, \ r_2 = -10i$

$y_1 = e^{r_1 x} = e^{10ix} = e^{i(10x)}$ $\qquad \boxed{e^{ix} = \cos x + i\sin x}$

$\boxed{y_1 = \cos(10x) + i\sin(10x)}$

$y_2 = e^{r_2 x} = e^{-10ix} = e^{i(-10x)}$

$y_2 = \cos(-10x) + i\sin(-10x)$

$\boxed{y_2 = \cos(10x) - i\sin(10x)}$

solutions have the same real part ( $\cos(10x)$ ) and

the same imaginary part ( $\sin(10x)$ ) but

opposite in sign $\rightarrow$ conjugate pairs general solution: $y = C_1 y_1 + C_2 y_2$

$$= C_1 \left[\cos(10x) + i\sin(10x)\right]$$
$$+ C_2 \left[\cos(10x) - i\sin(10x)\right]$$

the general solution is <u>real</u> but looks complex because $C$'s are also complex

rearrange:

$$y = \underbrace{(C_1 + C_2)}_{\text{call it } c_1 \text{ (real)}} \cos(10x) + \underbrace{i\,(C_1 - C_2)}_{\text{call it } c_2} \sin(10x)$$

$$\boxed{y = c_1 \underbrace{\cos(10x)}_{\substack{\text{real part} \\ \text{of } y_1 \text{ or } y_2}} + c_2 \underbrace{\sin(10x)}_{\substack{\text{imag. part} \\ \text{of } y_1 \text{ or } y_2}}}$$

**example** $y'' - 6y' + 25y = 0$

$$r^2 - 6r + 25 = 0$$

$$r = \frac{6 \pm \sqrt{36-100}}{2} = \frac{6 \pm \sqrt{-64}}{2} = \frac{6 \pm 8i}{2} = 3+4i,\ 3-4i$$

$$r_1 = 3+4i, \quad r_2 = 3-4i$$

as seen in previous example, we just need the real and imaginary parts of one solution ($y_1$)

$$y_1 = e^{r_1 x} = e^{(3+4i)x} = e^{3x} e^{i(4x)} \qquad e^{ix} = \cos x + i \sin x$$

$$= e^{3x}\left[\cos(4x) + i \sin(4x)\right]$$

real part: $e^{3x}\cos(4x)$ imag. part: $e^{3x}\sin(4x)$

linear combo of them form the <u>real</u> general solution

$$\boxed{y = C_1 e^{3x}\cos(4x) + C_2 e^{3x}\sin(4x)}$$

no $i$ in general solution!

example $6y^{(4)}+11y''+4y=0$

$$6r^4+11r^2+4=0$$

$$6u^2+11u+4=0 \qquad u=r^2$$

$$u=\frac{-11\pm\sqrt{121-96}}{12}=\frac{-11\pm 5}{12}=-\frac{4}{3},\ -\frac{1}{2}$$

$$r^2=-\frac{4}{3} \qquad\qquad r^2=-\frac{1}{2}$$

$$r=\underbrace{\frac{2}{\sqrt{3}}i,\ -\frac{2}{\sqrt{3}}i}_{\text{solve solutions as in prev. examples}},\ \underbrace{\frac{1}{\sqrt{2}}i,\ -\frac{1}{\sqrt{2}}i}_{\text{see prev. examples}}$$

real part:

$\cos\left(\frac{2}{\sqrt{3}}x\right)$

imag:

$\sin\left(\frac{2}{\sqrt{3}}x\right)$

real part: $\cos\left(\frac{1}{\sqrt{2}}x\right)$

imag part: $\sin\left(\frac{1}{\sqrt{2}}x\right)$

general solution:

$$\boxed{y = c_1 \cos\left(\frac{2}{\sqrt{3}}x\right) + c_2 \sin\left(\frac{2}{\sqrt{3}}x\right) + c_3 \cos\left(\frac{1}{\sqrt{2}}x\right) + c_4 \sin\left(\frac{1}{\sqrt{2}}x\right)}$$

<u>example</u> $y''' - 5y'' + 100y' - 500y = 0$

$$r^3 - 5r^2 + 100r - 500 = 0$$

$$r^2(r-5) + 100(r-5) = 0$$

$$(r-5)(r^2+100) = 0 \qquad r_1 = 10i, \quad r_2 = -10i, \quad r_3 = 5$$

$$\boxed{y = c_1 \cos(10x) + c_2 \sin(10x) + c_3 e^{5x}}$$