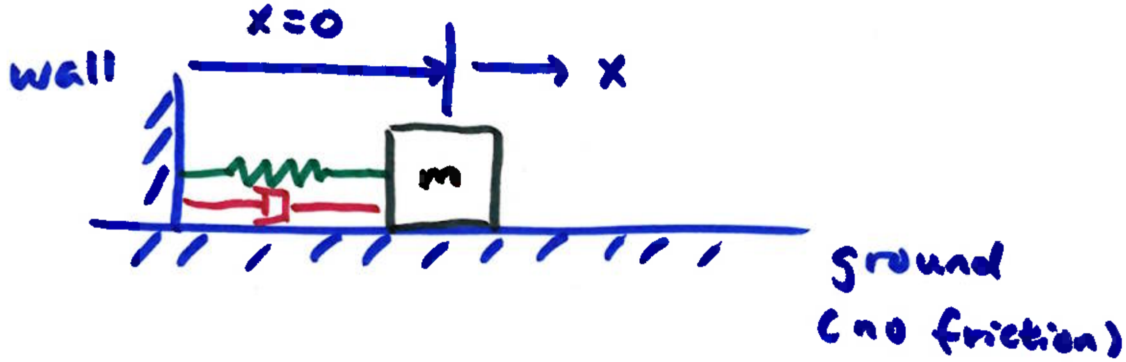
5.4 Mechanical Vibration

mass-spring-damper



mass m

spring constant k

damper (dashpot) damping constant c

$x=0$: equilibrium

spring wants to restore x to equilibrium

so provides a force of $F_s = -kx$

damper resists velocity $F_d = -cx'$

Newton's 2nd Law: $\sum F = mx''$

$$-kx - cx' = mx''$$

$$\boxed{mx'' + cx' + kx = 0}$$

2nd order linear constant coefficient eq.

<u>example</u> mass 8 kg, no damper

spring such that a force of 40 N stretches it by 5 cm.

Solve for the position of the mass if $x(0) = 0$, $x'(0) = 10$ m/s

$mx'' + cx' + kx = 0$ $\quad m = 8, \quad c = 0, \quad k =$ to be found

Hooke's Law: $F = kx$ change from equilibrium

$$40 = k(0.05)$$

5 cm in m

$$k = 800$$

$$8x'' + 800x = 0$$

$$x'' + 100x = 0$$

$$r^2 + 100 = 0 \qquad r = \pm 10i$$

$$\boxed{x(t) = C_1 \cos(10t) + C_2 \sin(10t)}$$

$x(0) = 0 \quad , \quad x'(0) = 10$

$x' = -10\, C_1 \sin(10t) + 10\, C_2 \cos(10t)$

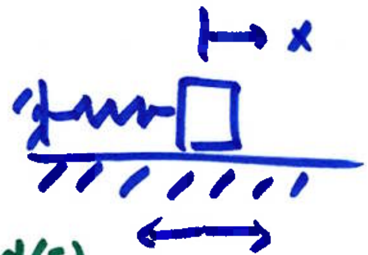
$x(0) = 0 \rightarrow 0 = C_1 \cos(0) + C_2 \sin(0) = C_1$

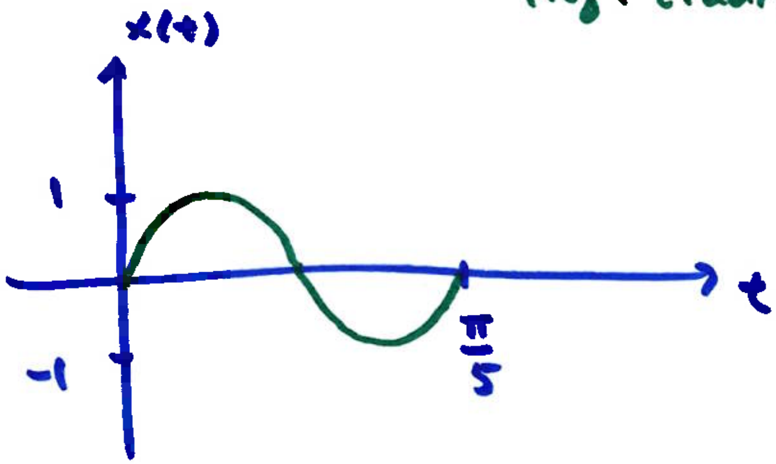
$x'(0) = 10 \rightarrow 10 = -10\, C_1 \sin(0) + 10\, C_2 \cos(0) = 10\, C_2$

$C_2 = 1$

$\boxed{x(t) = \sin(10t)}$

circular freq. (rad/s)





period : $\frac{2\pi}{10} = \frac{\pi}{5}$ seconds

circ. freq. : 10 rad/s

amplitude : 1

frequency : $\frac{1}{\text{period}} = \frac{5}{\pi}$ cycles per second (Hz)

Same setup, but now with $x(0)=2$ ($x'(0)=10$ as before)

$$x(t) = C_1\cos(10t) + C_2\sin(10t)$$

$$x'(t) = -10C_1\sin(10t) + 10C_2\cos(10t)$$

$$x(0)=2 \longrightarrow \cdots \longrightarrow C_1 = 2$$

$$x'(0)=10 \rightarrow \cdots \rightarrow C_2 = 1$$

now $\boxed{x(t) = 2\cos(10t) + \sin(10t)}$ amplitude?

omega phase shift

alternate form: $x(t) = C\cos(\omega t - \delta)$

ω: circular freq.

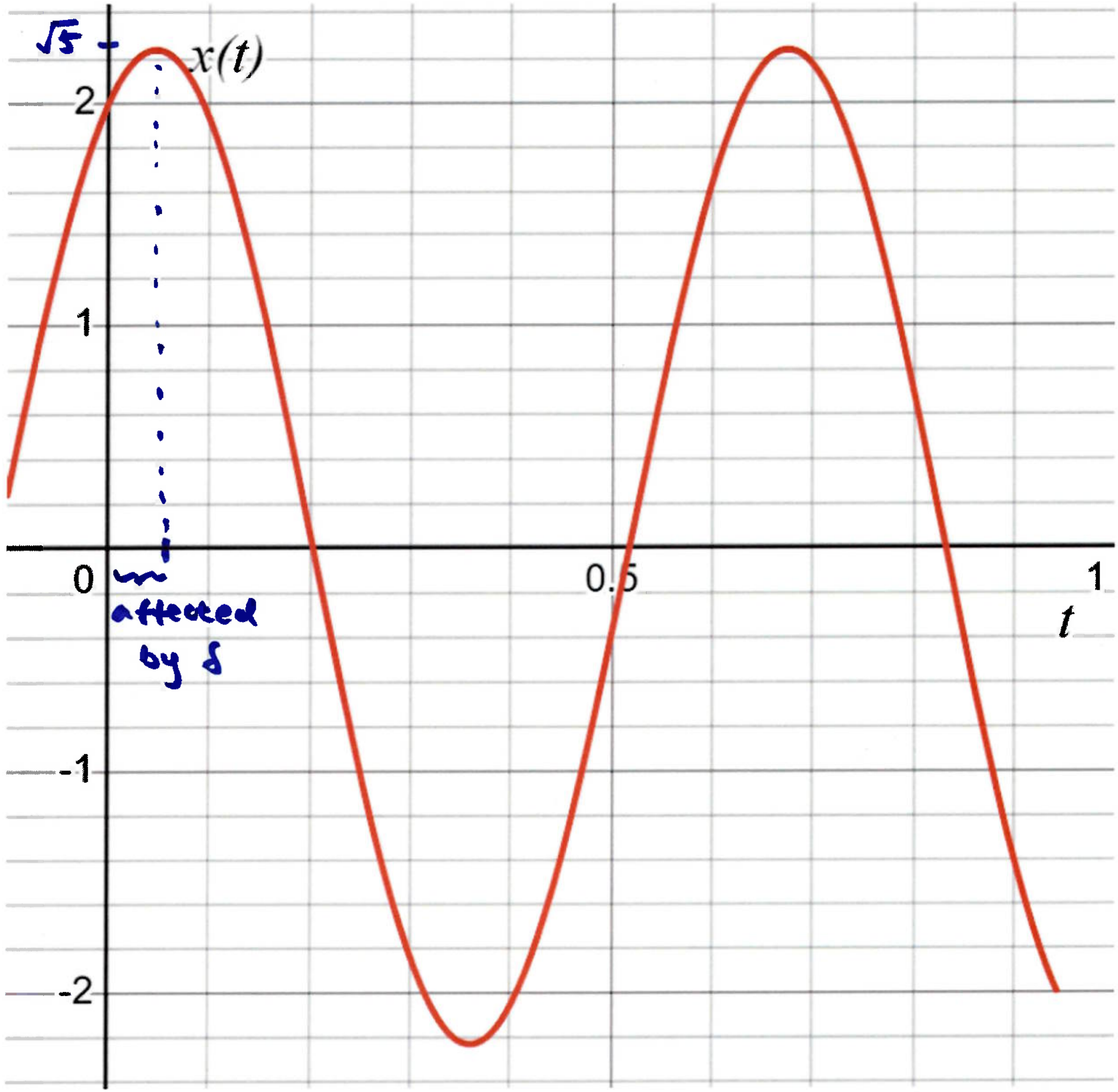
δ: phase shift

$$x(t) = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \delta)$$

using $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

we get

$$\boxed{C = \sqrt{A^2+B^2}} \qquad \boxed{\delta = \tan^{-1}\left(\frac{B}{A}\right)}$$



√5
x(t)
2
1
0
0.5
1
t
-1
-2
affected by δ

$$x(t) = \underset{A}{2}\cos(\underset{\omega}{10}t) + \underset{B}{1}\cdot\sin(10t)$$

$$\boxed{x(t) = \underset{C}{\sqrt{5}}\cos(10t - \underset{\delta}{0.464})}$$

amplitude $= \sqrt{5}$

now let's go back to $mx'' + cx' + kx = 0$

$m, c, k \neq 0$

$$mr^2 + cr + k = 0$$

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

$c^2 - 4km$ (discrimanent)

determines the type of roots

if $c^2 - 4km > 0$ $(c^2 > 4km)$

roots are real and distinct r_1, r_2

$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

strong damper $(c^2 > 4km)$ $\rightarrow$ <u>overdamped</u>

if $c^2 - 4km = 0$ $(c^2 = 4km)$

roots are real and repeated $r = r_1 = r_2$

$x(t) = c_1 e^{rt} + c_2 t e^{rt}$

damper tuned "just right" $\rightarrow$ <u>critically damped</u>

if $c^2 - 4km < 0$ $(c^2 < 4km)$

roots are complex $r = a \pm bi$

$x(t) = c_1 e^{at} \cos(bt) + c_2 e^{at} \sin(bt)$

weak damper $\rightarrow$ <u>underdamped</u>

the only case w/ oscillation

<u>example</u> $mx'' + cx' + kx = 0 \qquad x(0) = 0, \quad x'(0) = 8$

case 1: $c = 11, \ m = 10, \ k = 1$

overdamped $\quad x(t) = \frac{80}{9} e^{-\frac{1}{10}t} - \frac{80}{9} e^{-t}$

case 2: $c = 0, \ m = 10, \ k = 1$

undamped $\quad x(t) = 8\sqrt{10} \sin\left(\frac{1}{\sqrt{10}} t\right)$

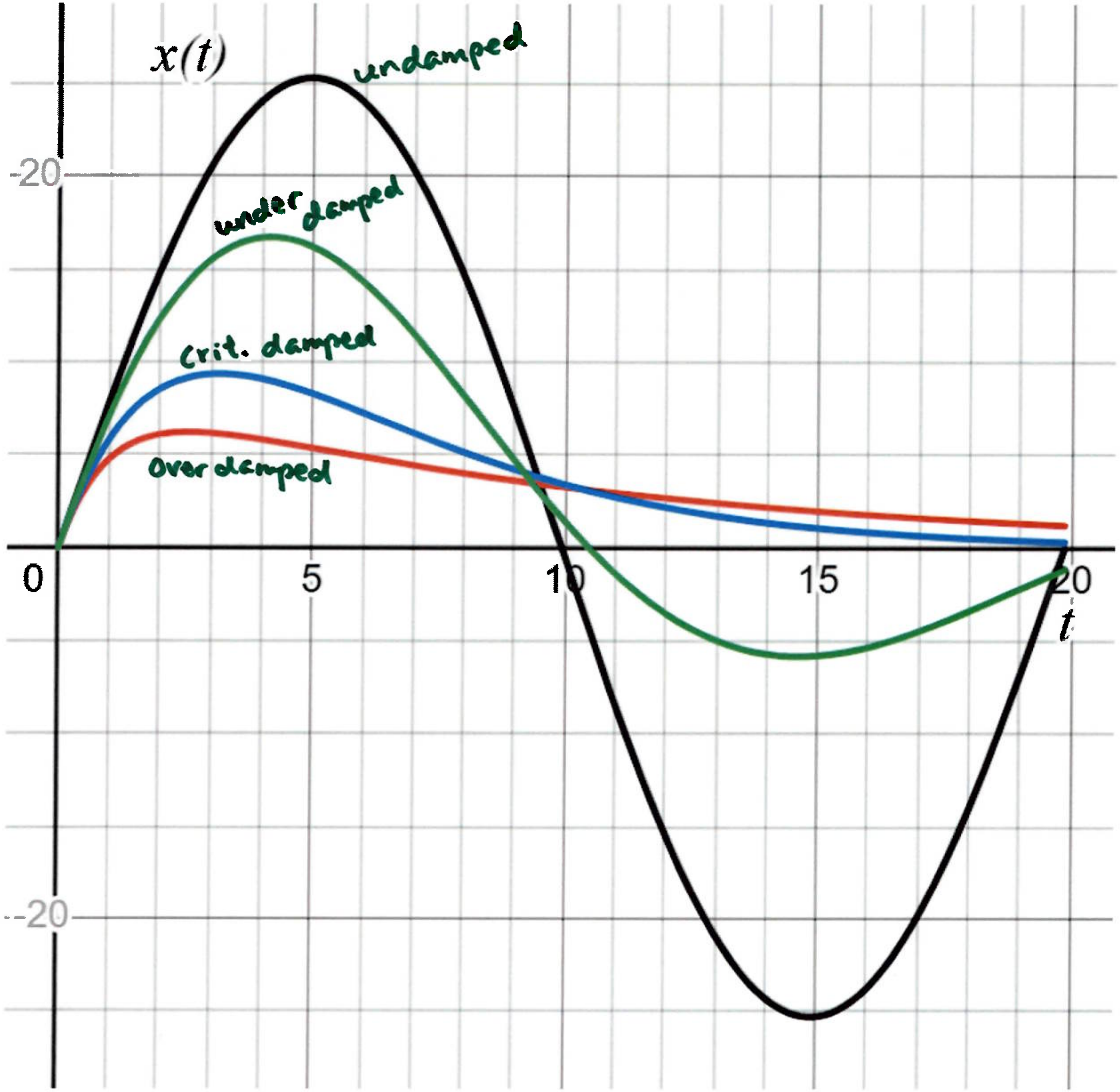
case 3: $c = \sqrt{40}, \ m = 10, \ k = 1$

crit. damped $\quad x(t) = 8t e^{-t/\sqrt{10}}$

case 4: $c = 2, \ m = 10, \ k = 1$

underdamped $\quad x(t) = \frac{80}{3} e^{-\frac{1}{10}t} \sin\left(\frac{3}{10} t\right)$

graphs on same plot



x(t)
undamped
underdamped
crit. damped
overdamped
20
0
5
10
15
20
t
-20