5.5 Nonhomogeneous Eqs.: Undetermined Coefficients

$$y'' + ay' + by = f(x)$$

if 0, homogeneous, we know how to solve
if nonzero, nonhomogeneous

because differential eq. above is linear, we can use the principle of superposition:

$$y = y_c + y_p$$

y_p → particular solution (due to $f(x)$ → forcing function)

y_c → solution to the homogeneous part $y'' + ay' + by = 0$ (complementary solution)

one method to find y_p is the method of undetermined coefficients

→ very effective if $f(x)$ is: polynomial
exponential
hyperbolic sine or hyperbolic cosine
cosine or sine

basic idea: y_p resembles $f(x)$

→ if, for example, $f(x)$ is a polynomial, then y_p is the same type of polynomial

<u>example</u> $y'' - 4y = 3x$

Solution: $y = y_c + y_p$

↳ solution to $y'' - 4y = 0$

$y'' - 4y = 0 \rightarrow r^2 - 4 = 0 \rightarrow r = \pm 2$

so, $y_c = c_1 e^{2x} + c_2 e^{-2x}$

particular solution y_p resembles $f(x) = 3x$

↳ first degree polynomial

assume a first degree polynomial y_p with <u>unknown coefficients</u>

$y_p = Ax + B$ $\quad$ A, B: undetermined coeffs.

even though $f(x) = 3x$, it does NOT necessarily mean $A = 3$, $B = 0$

y_p is a solution that satisfies $y'' - 4y = 3x$

$$\left.\begin{aligned} y_p &= Ax + B \\ y_p' &= A \\ y_p'' &= 0 \end{aligned}\right\} \text{ sub into}$$

$$0 - 4(Ax + B) = 3x$$

$$-4Ax - 4B = 3x + 0$$

$-4A = 3$ so $A = -\frac{3}{4}$

$-4B = 0$ so $B = 0$

so, $\boxed{y_p = -\frac{3}{4}x}$

general solution: $y = y_c + y_p$

$$\boxed{y = c_1 e^{2x} + c_2 e^{-2x} - \frac{3}{4}x}$$

<u>example</u> $y'' - y' - 2y = 3e^x$

$y = y_c + y_p$ $\quad$ y_c: solution to $y'' - y' - 2y = 0$

$$y_c = C_1 e^{-x} + C_2 e^{2x}$$

$f(x) = 3e^x$ is exponential e^x

so, we assume $y_p = Ae^x$

sub y_p into $y'' - y' - 2y = 3e^x$

$$\left.\begin{aligned} y_p &= Ae^x \\ y_p' &= Ae^x \\ y_p'' &= Ae^x \end{aligned}\right\} \quad Ae^x - Ae^x - 2Ae^x = 3e^x$$

$$-2A = 3$$

$$A = -\frac{3}{2}$$

$$y_p = -\frac{3}{2}e^x$$

$$\boxed{y = C_1 e^{-x} + C_2 e^{2x} - \frac{3}{2}e^x}$$

deriv. of exponential → exponential

deriv. of polynomial → polynomial

so assume y_p of the same form works ~~for~~ because of the same reason

we assume $y = e^{rx}$ in y_c

problem w/ sine and cosine : deriv. of sine is NOT sine

deriv. of cosine is NOT cosine

fix : if right side is cosine or sine or both

we assume y_p = has BOTH cosine and sine

because y_p' = has BOTH cosine and sine

the form is retained like w/ exponential and polynomial

example $y'' - y' - 2y = \cos x$

left side same as before: $y_c = c_1 e^{-x} + c_2 e^{2x}$

y_p needs to have BOTH $\cos x$ <u>AND</u> $\sin x$ even if only one appears in $f(x)$

$y_p = A\cos x + B\sin x$ need $\sin x$ even though it does not appear in $f(x)$

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

$$-A\cos x - B\sin x + A\sin x - B\cos x - 2A\cos x - 2B\sin x = \cos x$$

$$(-3A - B)\cos x + (-3B + A)\sin x = \cos x + 0\sin x$$

$$\left.\begin{aligned} -3A - B &= 1 \\ -3B + A &= 0 \end{aligned}\right\} \quad A = \frac{3}{10} \quad B = -\frac{1}{10} \quad y_p = \frac{3}{10}\cos x - \frac{1}{10}\sin x$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{2x} + \frac{3}{10}\cos x - \frac{1}{10}\sin x}$$

example $y'' - y' - 2y = xe^{3x}$

$y_c = c_1 e^{-x} + c_2 e^{2x}$

$f(x) = x e^{3x}$ ← exponential

↑ polynomial

$= xe^{3x} + 0 \cdot e^{3x} = (x+0)e^{3x}$

same form for y_p: $y_p = (Ax+B)e^{3x}$

$y_p = Axe^{3x} + Be^{3x}$

$y_p' = 3Axe^{3x} + Ae^{3x} + 3Be^{3x}$

$y_p'' = 9Axe^{3x} + 3Ae^{3x} + \cancel{Axe^{3x}} + 3Ae^{3x}$
$+ 9Be^{3x}$

sub into $y'' - y' - 2y = xe^{3x}$

$y_p = \ldots$

problem with undetermined coefficients: y_p duplicating y_c

<u>example</u> $y'' + 100y = \cos(10x)$

y_c: solution to $y'' + 100y = 0 \rightarrow r = \pm 10i$

$y_c = C_1 \cos(10x) + C_2 \sin(10x)$

$f(x) = \cos(10x)$

so, $y_p = A \underline{\cos(10x)} + B \underline{\sin(10x)}$

in y_c, not independent!

fix: throw in x's

proper $y_p = A\underline{\underline{x}} \cos(10x) + B\underline{\underline{x}} \sin(10x)$