5.5 Undetermined Coefficients (continued)

example $y'' - y = \cosh(x)$

$\cosh(x) = \dfrac{e^{x} + e^{-x}}{2}$

$\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$

$\dfrac{d}{dx}\cosh(x) = \sinh(x)$

$\dfrac{d}{dx}\sinh(x) = \cosh(x)$

Solution: $y = y_c + y_p$

y_c : solution to $y'' - y = 0$

$y_c = c_1 e^{x} + c_2 e^{-x}$

particular solution y_p : due to right side

option 1: rewrite $\cosh(x) = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$

guess form of $y_p = Ae^{x} + Be^{-x}$

now check if any of y_p duplicates y_c $\rightarrow$ both Ae^{x} and Be^{-x} duplicate y_c

fix: multiply the culprits by x

corrected y_p : $y_p = Axe^{x} + Bxe^{-x}$

plug y_p into $y'' - y = \cosh(x) = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$

$$y_p = Axe^{x} + Bxe^{-x}$$

$$y_p' = Ae^{x} + Axe^{x} + Be^{-x} - Bxe^{-x}$$

$$y_p'' = 2Ae^{x} + Axe^{x} - 2Be^{-x} + Bxe^{-x}$$

$$2Ae^{x} + \cancel{Axe^{x}} - 2Be^{-x} + \cancel{Bxe^{-x}} - \cancel{Axe^{x}} - \cancel{Bxe^{-x}} = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$$

$2A = \frac{1}{2}$ so $A = \frac{1}{4}$

$-2B = \frac{1}{2}$ so $B = -\frac{1}{4}$

so, $\boxed{y_p = \frac{1}{4}xe^{x} - \frac{1}{4}xe^{-x}}$

$$y'' - y = \cosh(x)$$

option 2: work w/ hyperbolic functions

$$y_c = c_1 e^{x} + c_2 e^{-x}$$

from definitions of $\cosh(x)$ and $\sinh(x)$ we can get

$$e^{x} = \cosh(x) + \sinh(x)$$

$$e^{-x} = \cosh(x) - \sinh(x)$$

$$y_c = C_1\cosh(x) + C_1\sinh(x) + C_2\cosh(x) - C_2\sinh(x)$$

$$= (C_1 + C_2)\cosh(x) + (C_1 - C_2)\sinh(x)$$

abuse/reuse C's

$$y_c = C_1\cosh(x) + C_2\sinh(x)$$

right side: $\cosh(x)$

guess y_p: $y_p = A\cosh(x) + B\sinh(x)$ just like w/ $\cos(x)$ and $\sin(x)$

$$\cancel{y_p' = A\sinh(x) + B\cosh(x)}$$

$$\cancel{y_p'' = A\cosh(x) + B\sinh(x)}$$

check duplication: both $A\cosh(x)$ and $B\sinh(x)$ are duplicating y_c

fix: throw in x's

Corrected y_p: $y_p = Ax\cosh(x) + Bx\sinh(x)$

$$y_p' = Ax\sinh(x) + A\cosh(x) + B\sinh(x) + Bx\cosh(x)$$

$$y_p'' = 2A\sinh(x) + Ax\cosh(x) + Bx\sinh(x) + 2B\cosh(x)$$

Sub into $y'' - y = \cosh(x)$

$\vdots$

$A = 0, \quad B = +\frac{1}{2}$ so, $\boxed{y_p = \frac{1}{2} x \sinh(x)}$

<u>example</u> $y^{(5)} + 2y^{(4)} - y = 3$

find y_p.

y_c : from roots of $r^5 + 2r^4 - 1 = 0$

how to factor this ??

look at form of y_p first.

right side: $3 \rightarrow$ form of y_p is $y_p = A$

does it duplicate any part of y_c ?

if $y_p = A$ were to duplicate any part of y_c,

one of the roots of $r^5 + 2r^4 - 1 = 0$ must be 0

because $y = e^{0x} = 1 \rightarrow$ constant duplicated by $y_p = A$

is $r=0$ a root of $r^5+2r^4-1=0$? NO.

so no risk of duplication, so $y_p = A$ is ok.

$y^{(5)}+2y^{(4)}-y=3$ sub in $y_p = A$

$y_p' = 0$

$y_p'' = 0$

$\vdots$

$y_p^{(n)} = 0$

$-A = 3$

$A = -3$

so, $\boxed{y_p = -3}$

<u>example</u> Find the correct form of y_p w/ solving the constants

$$y'' + y = x\cos x$$

$$y_c = C_1\cos x + C_2 \sin x$$

right side: $x \cos x$

1st-degree polynomial

cosx and sinx appear in y_p together

initial guess of y_p:

$$y_p = (Ax+B)\cos x + (Cx+D)\sin x$$

$$y_p = Ax\cos x + B\cos x + Cx\sin x + D\sin x$$

duplication? yes. ~~$Ax\cos x$~~ $B\cos x$ and $D\sin x$ duplicate y_c

fix: throw in x → $Bx\cos x$ doesn't duplicate y_c
but duplicates $Ax\cos x$ in y_p

fix: throw in another x

corrected y_p:

$$y_p = Ax\cos x + Bx^2\cos x + Cx\sin x + Dx^2\sin x$$

$y_p' = \dots$

$y_p'' = \dots$

Sub into $y'' + y = x\cos x$

$A = \frac{1}{4}, \quad B = 0, \quad C = 0, \quad D = \frac{1}{4}$

example $y^{(5)} + 2y^{(3)} + 2y'' = 3x^2 - 1$

form of y_p only

initial guess: $y_p = Ax^2 + Bx + C$

if this were to duplicate y_c, at least one root is 0

$$r^5 + 2r^3 + 2r^2 = r^2(\underbrace{r^3 + 2r + 2})$$

0 , 0

exponential, cosine, sine, but NOT 0

$y_1 = 1 \quad (e^{0x})$

$y_2 = x \quad$ (second e^{0x})

$y_3 =$

$y_4 =$

$y_5 =$

$y_p = Ax^2 + Bx + C$

duplicates duplicates

throw in one x: duplicates still

throw in two x: ok!

$$y_p = x^2(Ax^2 + Bx + C)$$

next time: another method → Variation of Parameters