5.5 The Method of Variation of Parameters

another method to solve $y'' + ay' + by = f(x)$

undetermined coefficients only works if $f(x)$ is a function that remains the same type after derivatives

good: polynomials, exponentials, sine + cosine } remain the same form after derivatives

bad: for example, $f(x) = \tan x$

$f'(x) = \sec^2 x$ NOT $\tan x$ anymore

another: $f(x) = \sec x$

$f'(x) = \sec x \tan x$ form changed

variation of parameters can (in principle) deal with any $f(x)$

→ first discovered / discussed by Euler, refined by Lagrange

Variation of Parameters

$y'' + ay' + by = f(x)$

if y_1 and y_2 are the two linearly independent solutions to $y'' + ay' + by = 0$

then the general solution to $y'' + ay' + by = f(x)$ is $\boxed{y = u_1 y_1 + u_2 y_2}$ → equivalent to $y = y_c + y_p$

u_1, u_2 are functions of x (they ~~are~~ are the "parameters")

what are u_1, u_2?

$y = u_1 y_1 + u_2 y_2$ is a solution to $y'' + ay' + by = f(x)$

Sub $y = u_1 y_1 + u_2 y_2$ into →

$y' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$

to avoid having to deal with u_1'' and u_2'', we will set

$u_1' y_1 + u_2' y_2 = 0$ → ok to do because we have two unknowns and one equation

$$\left.\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ y' &= u_1 y_1' + u_2 y_2' \\ y'' &= u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' \end{aligned}\right\}$$ sub into $y'' + ay' + by = f(x)$

$$u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' + a u_1 y_1' + a u_2 y_2' + b u_1 y_1 + b u_2 y_2 = f(x)$$

collect terms

$$u_1 \underbrace{(y_1'' + a y_1' + b y_1)}_{\substack{0 \text{ because } y_1 \\ \text{is a solution} \\ \text{to } y''+ay'+by=0}} + u_2 \underbrace{(y_2'' + a y_2' + b y_2)}_{0 \text{ same reason}} + u_1' y_1' + u_2' y_2' = f(x)$$

to find u_1 and u_2, we solve

$$\boxed{\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f(x) \end{aligned}}$$

gen solution: $y = u_1 y_1 + u_2 y_2$

<u>example</u> $y'' - y' - 2y = 3e^{2x}$

undetermined is usable, but we will use variation here

first, find y_1, y_2

$y'' - y' - 2y = 0 \longrightarrow \quad y_1 = e^{2x} \quad y_2 = e^{-x}$

if undetermined is used, we need to adjust y_p to avoid duplication

variation doesn't care!

solve:

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = f(x)$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$\begin{bmatrix} e^{2x} & e^{-x} & 0 \\ 2e^{2x} & -e^{-x} & 3e^{2x} \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} e^{2x} & e^{-x} & 0 \\ 0 & -3e^{-x} & 3e^{2x} \end{bmatrix}$$

row 2: $-3e^{-x}u_2' = 3e^{2x}$

$$u_2' = -e^{3x}$$

row 1: $e^{2x}u_1' + e^{-x}u_2' = 0$

$$e^{2x}u_1' = -e^{-x}u_2' = e^{2x}$$

$$u_1' = 1$$

so, $u_1 = \int u_1'\,dx = x + c_1$

$$u_2 = \int u_2'\,dx = -\frac{1}{3}e^{3x} + c_2$$

gen. solution is $y = u_1y_1 + u_2y_2$

$$y = (x+c_1)e^{2x} + \left(-\frac{1}{3}e^{3x} + c_2\right)e^{-x}$$

$$= \underline{c_1e^{2x}} + c_2e^{-x} + xe^{2x} \underline{- \frac{1}{3}e^{2x}}$$

add to

$$\boxed{y = \underbrace{c_1e^{2x} + c_2e^{-x}}_{y_c} + \underbrace{xe^{2x}}_{y_p}}$$

example $y'' + y = \sec x$ undetermined is unable to solve

$y'' + y = 0 \rightarrow y_1 = \cos x \quad y_2 = \sin x$

solve $u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = f(x)$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

let's solve this the matrix way: $A\vec{x} = \vec{b} \rightarrow \vec{x} = A^{-1}\vec{b}$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f(x) \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$\rightarrow$ Wronskian of y_1 and y_2
NOT zero if y_1 and y_2
are linearly independent

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W(x)} \begin{bmatrix} -y_2 f(x) \\ y_1 f(x) \end{bmatrix}$$

$$u_1' = \frac{-y_2 f(x)}{W(x)} \qquad u_2' = \frac{y_1 f(x)}{W(x)}$$

here, $y_1 = \cos x \quad y_2 = \sin x \quad f(x) = \sec x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$ (generally a function of x)

$$u_1' = \frac{-\sin x \cdot \sec x}{1} = -\frac{\sin x}{\cos x}$$

$$u_1 = \ln|\cos x| + C_1$$

$$u_2' = \frac{\cos x \cdot \sec x}{1} = 1$$

$$u_2 = x + C_2$$

$$y = u_1 y_1 + u_2 y_2 = (C_1 + \ln|\cos x|)\cos x + (x + C_2)\sin x$$

$$= \boxed{C_1 \cos x + C_2 \sin x + \cos x \ln|\cos x| + x \sin x}$$