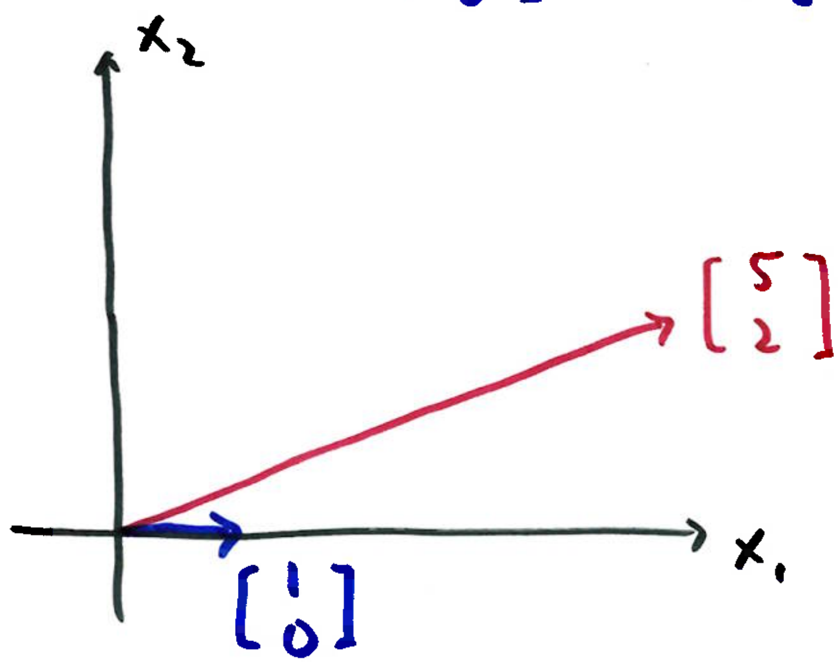
6.1 Eigenvalues and Eigenvectors

when we multiply a matrix A and a vector $\vec{x}$, we can interpret it as a transformation of vector $\vec{x}$

for example, $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$

if $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A\vec{x} = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

matrix transformed $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ into $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$



$A\vec{x}$ changes the direction and magnitude of $\vec{x}$

most matrices have special vectors that preserve their directions

$$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad A\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

keeps direction
magnitude changes by
factor of 1

$$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad A\vec{x} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

keeps direction
magnitude changes by
a factor of 5

these vectors that keep their directions after transformation
are called the eigenvectors of the matrix

($n \times n$ matrix has n of these)

the factors by which their magnitudes change are called
~~then~~ the eigenvalues

in the example above, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector w/ associate eigenvalue 1

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ " " " " 5 if $\vec{v}$ is an eigenvector of A with the associated eigenvalue λ

then $\boxed{A\vec{v} = \lambda\vec{v}}$

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$\boxed{(A - \lambda I)\,\vec{v} = \vec{0}}$$

identity matrix

homogeneous equation

to have nontrivial $\vec{v}$ (eigenvector cannot be $\vec{0}$)

$$\rightarrow \quad \boxed{\det(A - \lambda I) = 0}$$

solve this for λ, then sub into

$(A - \lambda I)\vec{v} = \vec{0}$ for $\vec{v}$

example

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$$

first, solve $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-\lambda & 4 \\ -3 & -1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 4 \\ -3 & -1-\lambda \end{vmatrix} = 0$$

$$= (7-\lambda)(-1-\lambda) - (-3)(4) = 0$$

$$\boxed{\lambda^2 - 6\lambda + 5 = 0}$$ characteristic equation

n^{th}-order for $n \times n \rightarrow n$ λ's

$$(\lambda - 5)(\lambda - 1) = 0$$

$$\boxed{\lambda = 1, \quad \lambda = 5}$$ eigenvalues of A

now solve $(A - \lambda I)\vec{v} = \vec{0}$ using these λ's for $\vec{v}$'s

$\underline{\lambda = 1}$ $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc} 6 & 4 & 0 \\ -3 & -2 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $x_2 = \text{free} = r$

$3x_1 + 2x_2 = 0$

$x_1 = -\frac{2}{3} x_2 = -\frac{2}{3} r$

so, $\vec{v} = \begin{bmatrix} -\frac{2}{3} r \\ r \end{bmatrix}$

choose any $r \neq 0$ to make $\vec{v}$ convenient

here, choose $r = -3$

$$\boxed{\vec{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad \lambda = 1}$$

$\lambda = 5 \quad (A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 2 & 4 & 0 \\ -3 & -6 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = r \quad x_1 + 2x_2 = 0 \quad x_1 = -2x_2 = -2r$

Choose $r = 1$ $\quad \boxed{\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \ \lambda = 5}$

spans a space called <u>eigenspace</u>

here, this eigenspace has a dimension of 1 (one basis vector)

<u>example</u>

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

cofactor expansion along column 2

$$(1-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(4-\lambda)(1-\lambda)+2\right] = 0$$

$$(1-\lambda)(\lambda^2-5\lambda+6) = 0$$

$$(1-\lambda)(\lambda-2)(\lambda-3) = 0$$

$$\boxed{\lambda = 1,\ 2,\ 3}$$

$\underline{\lambda = 1}$ $\quad (A-\lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_2 = r \quad x_3 = 0 \quad x_1 = 0$

$$\vec{v} = \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix} \quad \text{choose } r=1 \quad \boxed{\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \lambda = 1}$$

$\underline{\lambda = 2}$ $\cdots$ $\vec{v} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

$\underline{\lambda = 3}$ $\cdots$ $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

again, each eigenspace has dimension of 1

if A is triangular or diagonal, the λ's are the main diagonal elements

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$ triangular

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 4-\lambda & 5 \\ 0 & 6-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda)(6-\lambda) = 0$$

$\lambda = 1, 4, 6 \rightarrow$ diagonal numbers eigenvalues and eigenvectors can be complex

example $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 4 = 0 \qquad \boxed{\lambda = 2i,\ -2i}$$

always conjugate pairs

$\lambda = 2i$ $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} -2i & 1 & 0 \\ -4 & -2i & 0 \end{bmatrix}$$

$(-2i)(2i) = -4i^2 = 4$

$$\xrightarrow{(2i)R_1 + R_2} \begin{bmatrix} -2i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = r \qquad -2i\,x_1 + x_2 = 0$

$$x_1 = \frac{1}{2i} r$$

$\vec{v} = \begin{bmatrix} \frac{1}{2i} r \\ r \end{bmatrix}$ choose $r = -2i$ $\boxed{\vec{v} = \begin{bmatrix} -1 \\ -2i \end{bmatrix} \quad \lambda = 2i}$

$\underline{\lambda = -2i}$ unsurprisingly, $\vec{v}$ is the conjugate of the other one $\rightarrow$ $\boxed{\vec{v} = \begin{bmatrix} -1 \\ 2i \end{bmatrix} \quad \lambda = -2i}$