7.1 First-order systems

system of 1st-order differential eqs.

$$x' = f(x, y, t)$$
$$y' = g(x, y, t)$$

system of two eqs.

for example, $x' = y$, $y' = -x$ } each equation can depend on other dependent variables (x, y) as well as independent (t)

notice the equations are coupled → x' depends on y, need to know y to solve for x (vice versa in other eq.)

one way to solve (we'll look at other ways in rest of ch. 7) is to convert into a second-order equation

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

$$\hookrightarrow x'' = y' = -x \longrightarrow x'' + x = 0$$

$$r^2 + 1 = 0 \quad r = \pm i$$

$$\boxed{x(t) = C_1 \cos t + C_2 \sin t}$$

to find y, use $x' = y \rightarrow$ $\boxed{y(t) = -C_1 \sin t + C_2 \cos t}$

in general, n 1st-order equations can be turned into one n^{th}-order equation.

and this goes the other way, too.

one n^{th}-order $\longrightarrow$ n 1st-order eqs. system

example

$$x'' + 3x' + 7x = t^2$$

2nd-order → system of 2 1st-order

define two variables to represent x and x'

all dependent variables below the highest order

let $x_1 = x$ and $x_2 = x'$

$$x_1' = x' = x_2 \longrightarrow \boxed{x_1' = x_2}$$

first eq. in system

now we need one for x_2: $x_2' = g(x_1, x_2, t)$

$$x'' = -3x' - 7x + t^2$$

solve for highest order

x_2' x_2 x_1

$$\boxed{x_2' = -3x_2 - 7x_1 + t^2}$$

second eq. in the system

example $x^{(4)} + 6x''' - 3x'' + x' + 10x = \cos 3t$

4^{th}-order $\longleftrightarrow$ 4 1st-order

need four variables : $\underbrace{x''', x'', x', x}$

dep. variables below highest order

let $x_1 = x$

$x_2 = x'$

$x_3 = x''$

$x_4 = x'''$

notice $\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \end{aligned}}$

last one: from $x^{(4)} = -6(x''') + 3(x'') - (x') - 10(x) + \cos 3t$

x_4' x_4 x_3 x_2 x_1

$$\boxed{x_4' = -6x_4 + 3x_3 - x_2 - 10x_1 + \cos 3t}$$

system of two 2nd-order → system of four 1st-order

Example

$$x'' - 5x' - 4x + 6y = 0$$
$$y'' + 6y' + 5x + 5y = 0$$

each 2nd-order → two 1st-order

let $x_1 = x \quad x_2 = x'$

let $y_1 = y \quad y_2 = y'$

write differential eqs. for x_1, x_2, y_1, y_2

$$x_1' = x_2$$
$$x_2' = 5x_2 + 4x_1 - 6y_1$$
$$y_1' = y_2$$
$$y_2' = -6y_2 - 5x_1 - 5y_1$$

$$\begin{bmatrix} x_1' \\ x_2' \\ y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & 5 & -6 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 0 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \rightarrow \vec{x}' = A\vec{x}$$