7.2 Matrices and Linear Systems

last time: n^{th}-order $\longleftrightarrow$ n 1^{st}-order system

for example, $t^2 x'' + t x' + (t^2 - 1) x = e^t$

$$x'' + \frac{1}{t} x' + \frac{t^2 - 1}{t^2} x = \frac{e^t}{t^2}$$

define two variables: $x_1 = x$, $x_2 = x'$

up to one below highest deriv.

$x_1' = x_2$ equation 1

from differential eq. at top

$$x'' = -\frac{1}{t} x' - \frac{t^2-1}{t^2} x + \frac{e^t}{t^2}$$

$\downarrow$

$$x_2' = -\frac{1}{t} x_2 - \frac{t^2-1}{t^2} x_1 + \frac{e^t}{t^2} \qquad \text{eq. 2}$$

rewrite as a matrix equation

$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_{\vec{x}\,'} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{t^2-1}{t^2} & -\frac{1}{t} \end{bmatrix}}_{\substack{p(t) \\ \text{coeff. matrix}}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ \frac{e^t}{t^2} \end{bmatrix}}_{\vec{f}(t)}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\vec{f}(t)$: vector function of t w/o dependent variables

another example

$$x' = 8x - 2y + z + t$$
$$y' = x - 4z + t^2$$
$$z' = 5y - 2z + t^3$$

write as $\vec{x}\,' = p(t)\vec{x} + \vec{f}(t)$

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \vec{x}\,' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

as matrix eq.

$$\vec{x}\,' = \begin{bmatrix} 8 & -2 & 1 \\ 1 & 0 & -4 \\ 0 & 5 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$$

many properties of $\vec{x}\,' = p(t)\vec{x} + \vec{f}(t)$ are the same as the scalar case.

if $\vec{f}(t) = \vec{0}$, $\vec{x}\,' = p\vec{x}$ is a homogeneous system

a solution is a function $\vec{x}(t)$ that satisfies the matrix eq.

if $p(t)$ is an $n \times n$ ~~by~~ matrix, there are n linearly independent solutions $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$

because $n \times n \leftrightarrow n^{th}$-order scalar eq.

the general solution is $\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \dots + c_n \vec{x}_n$

the **Wronskian** of the solutions $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ is

$$W = \begin{vmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{vmatrix}$$ solutions are columns

if $W \neq 0$ on some interval, then the $\vec{x}$'s are linearly independent on that interval

example Verify that $\vec{x}_1 = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{x}_2 = e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ are solutions of $\vec{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\vec{x}$

first, check if they satisfy $\vec{x}' = P\vec{x}$

$$\vec{x}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} \qquad \vec{x}_1' = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \end{bmatrix} = 2e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

plug into $\vec{x}' = P\vec{x}$

$$\begin{bmatrix} 2e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} \quad ?$$

$$= \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{yes}$$

Similarly, we can show that $\vec{x}_2 = \begin{bmatrix} e^{-2t} \\ 5e^{-2t} \end{bmatrix}$ is a solution

Wronskian of $\vec{x}_1$, $\vec{x}_2$

$$W = \begin{vmatrix} \vec{x}_1 & \vec{x}_2 \end{vmatrix} = \begin{vmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{vmatrix} = 4 \neq 0 \text{ so}$$

$\vec{x}_1$, $\vec{x}_2$ are linearly indp

general solution:

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

c's depend on initial conditions (vectors)

$\vec{x}\,' = p(t)\,\vec{x} + \vec{f}(t)$ nonhomogeneous if $\vec{f} \neq \vec{0}$

Solution: $\vec{x} = \vec{x}_c + \vec{x}_p$

$\vec{x}_p$ → particular (due to $\vec{f}$)

$\vec{x}_c$ → complementary (solution of $\vec{x}\,' = p\,\vec{x}$)

Undetermined Coefficients

$$\underbrace{y'' - y}_{} = x + x\cos(x) + x^2 e^{2x}$$

$y'' - y = 0 \rightarrow y_c = c_1 e^{x} + c_2 e^{-x}$

no danger of y_p duplicating it

particular: $x + x\underbrace{\cos(x)}_{} + \underbrace{x^2 e^{2x}}_{}$

1st-degree polynomial

$Ax + B$

1st-deg polynomial attached to cosine need sine

$((Cx + D)\cos(x) + (Ex + F)\sin(x)$

$\hookrightarrow$ 2nd-deg attached to exponential

$(Gx^2 + Hx + I)e^{2x}$

$$y_p = Ax + B + ((Cx + D)\cos x + (Ex + F)\sin x + (Gx^2 + Hx + I)e^{2x}$$

if right side changed to $y'' + y$

$$y_c = C_1 \cos x + C_2 \sin x$$

$$y_p = Ax + B + (Cx + \underline{D)\cos x} + (Ex + \underline{F)\sin x} + (Gx^2 + Hx + I)e^{2x}$$

duplicating y_c

x alone doesn't solve the problem

because $Dx\cos x$ duplicates $Cx\cos x$

so, need x^2 to ALL cos/sin terms

$$y_p = \cdots + x^2(Cx+D)\cos x + x^2(Ex+F)\sin x + \cdots$$

no extra x

because NOT duplicating y_c

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = f$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & f \end{bmatrix} \text{ ... reduce, find } u_1', u_2'$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1}}_{\frac{1}{\underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}_{W}} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}} \begin{bmatrix} 0 \\ f \end{bmatrix}$$

equivalent