7.3 The Eigenvalue Method for Linear Systems

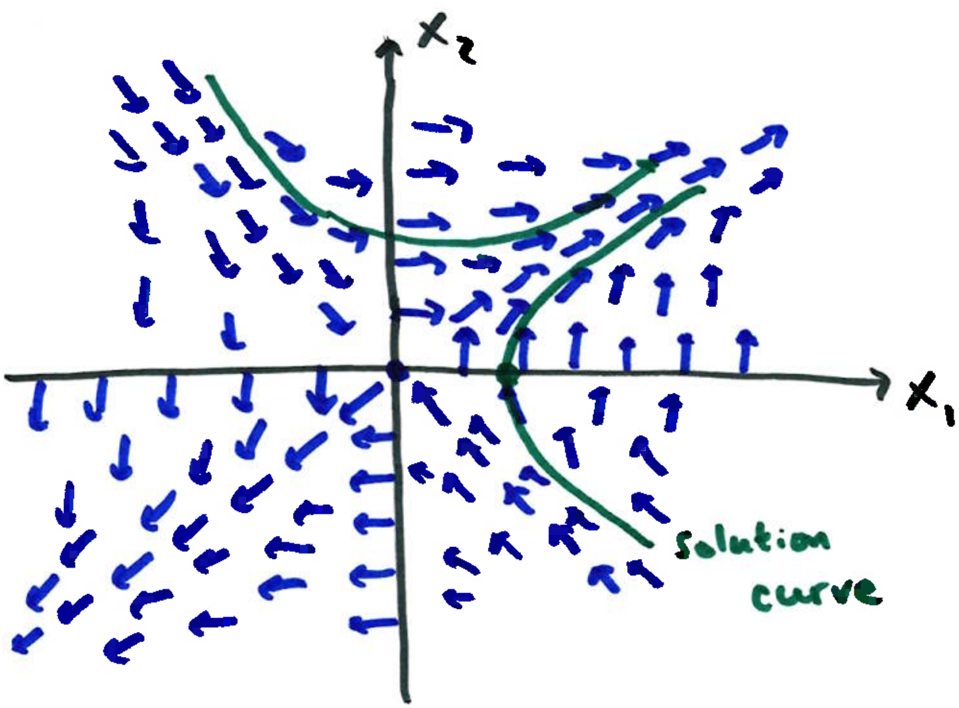
solve $\vec{x}' = A\vec{x}$ A: constant matrix

first method: graphical (equivalent of slope field)

<u>example</u> $\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{x}$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\vec{x}' \rightarrow$ vector tangent the solution curve $\vec{x}(t)$ at a given point

collect some $\vec{x}'$ at selected points



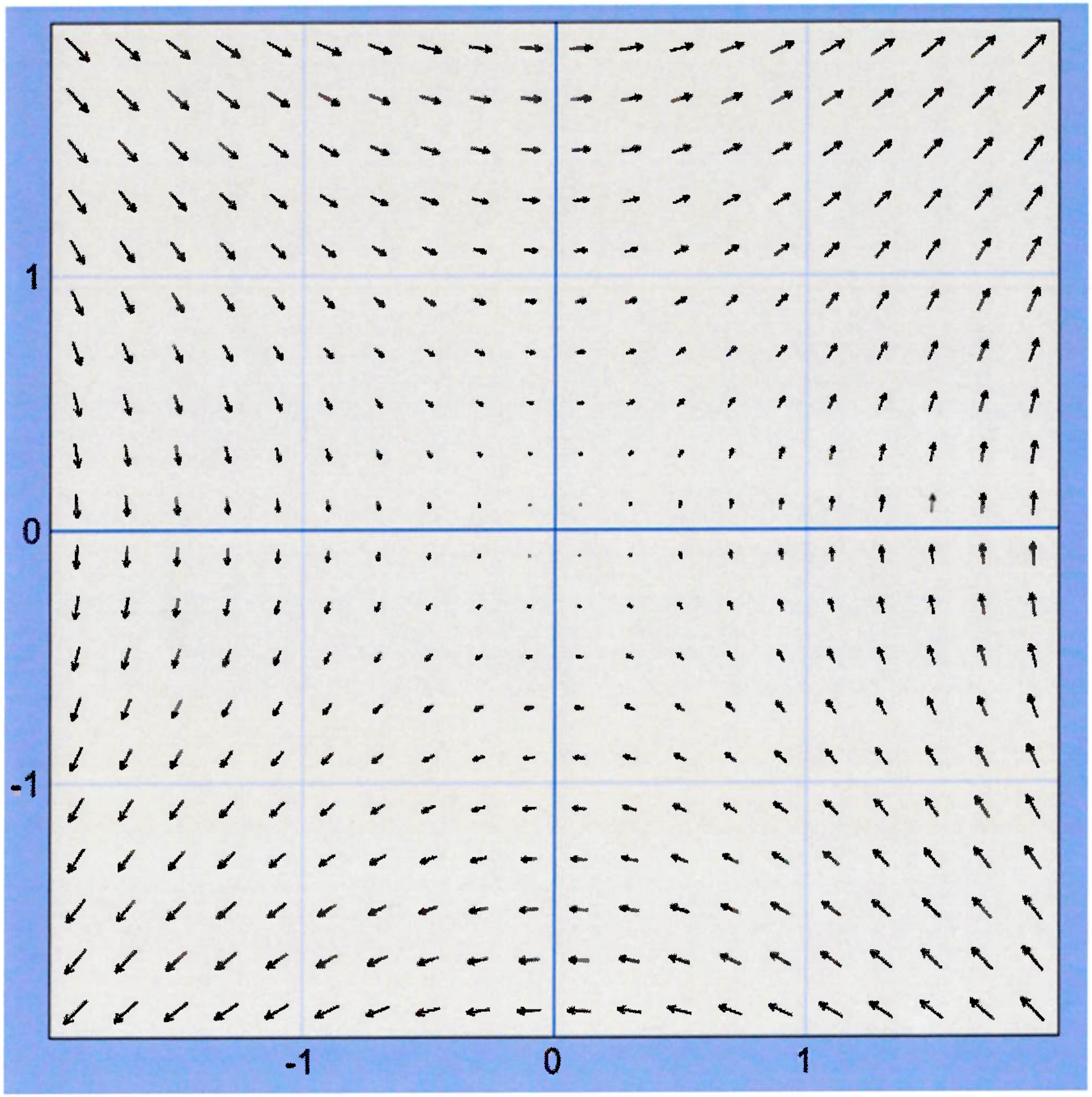
pick $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$


1
0
-1
-1
0
1

no precision from slope field

let's solve it completely

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}$$

convert back to scalar eq:

$$x_1' = x_2$$
$$x_2' = x_1$$
$$x_1'' = x_2' = x_1$$

$$x_1'' - x_1 = 0 \qquad r^2 - 1 = 0 \qquad r = \pm 1$$

$$\boxed{\begin{aligned} x_1 &= c_1 e^{t} + c_2 e^{-t} \\ x_2 &= c_1 e^{t} - c_2 e^{-t} \end{aligned}} \qquad (x_2 = x_1')$$

we don't want to convert all the time

let's look for pattern in the solution and connect to matrix A

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underset{e^{at},\ a=1}{c_1 e^{t}} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{vector}} + \underset{e^{at},\ a=-1}{c_2 e^{-t}} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{vector}}$$

value 1, vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

value -1, vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

could these be the eigenvalue/eigenvector pairs of A?

let's check:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \qquad \lambda^2 - 1 = 0$

$\lambda = 1, -1$

$\underline{\lambda = 1}$ solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \cdots \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = -1}$ solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \cdots \quad \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so, they are indeed the eigenvalue/eigenvector pairs of A

Solution to $\vec{x}\,' = A\vec{x}$ is

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v_1} + c_2 e^{\lambda_2 t}\vec{v_2}$$

when $\lambda_1, \vec{v_1}$ and $\lambda_2, \vec{v_2}$ are the eigenvalue/eigenvector pairs

<u>example</u> $x_1' = x_1 + 2x_2$

$x_2' = 2x_1 + x_2$

$\vec{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x}$ find $\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$

$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$ $(1-\lambda)^2 - 4 = 0$

$1-\lambda = \pm 2$ $\lambda = -1$ $\lambda = 3$

<u>$\lambda = -1$</u> $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$(A - \lambda I)\vec{v} = \vec{0}$

<u>$\lambda = 3$</u> $\begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

solution

$$\boxed{\vec{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

if A is 3×3, $\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + c_3 e^{\lambda_3 t}\vec{v}_3$

<u>example</u>

$$x_1' = 4x_1 + 3x_2 + 3x_3$$
$$x_2' = -8x_1 - 7x_2 - 3x_3$$
$$x_3' = 8x_1 + 8x_2 + 4x_3$$

$$\vec{x}' = \begin{bmatrix} 4 & 3 & 3 \\ -8 & -7 & -3 \\ 8 & 8 & 4 \end{bmatrix}\vec{x}$$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 4-\lambda & 3 & 3 \\ -8 & -7-\lambda & -3 \\ 8 & 8 & 4-\lambda \end{vmatrix} = 0$$

$\lambda = -4,\ 4,\ 1$

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

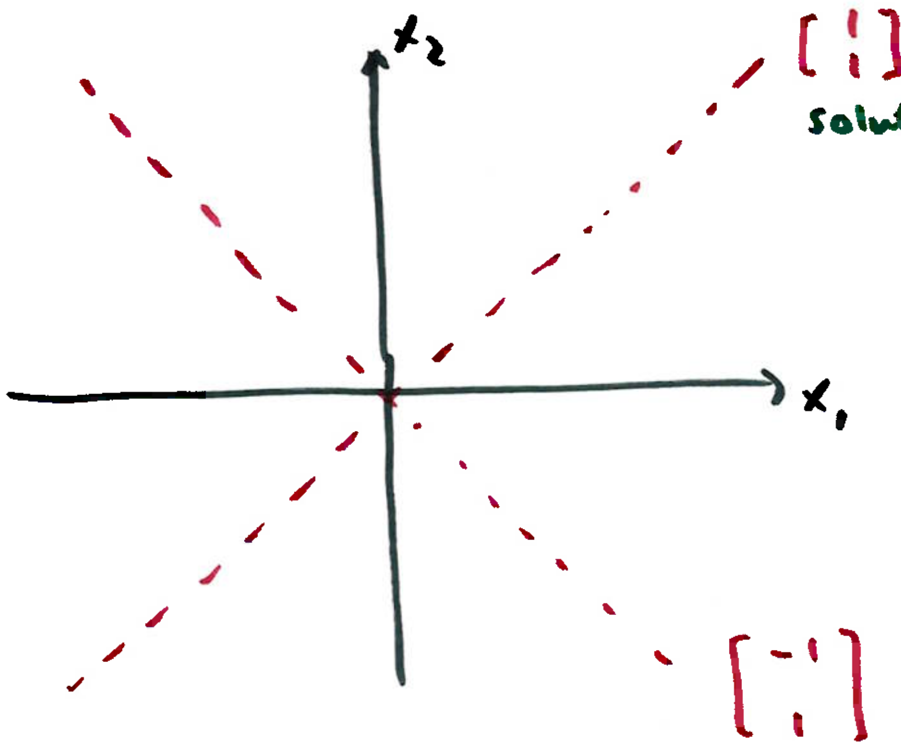
$$\boxed{\vec{x} = c_1 e^{-4t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}$$

back to $\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}$

solution: $\vec{x} = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lim_{t \to \infty} \vec{x} \to c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to$ parallel to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (eigenvector)

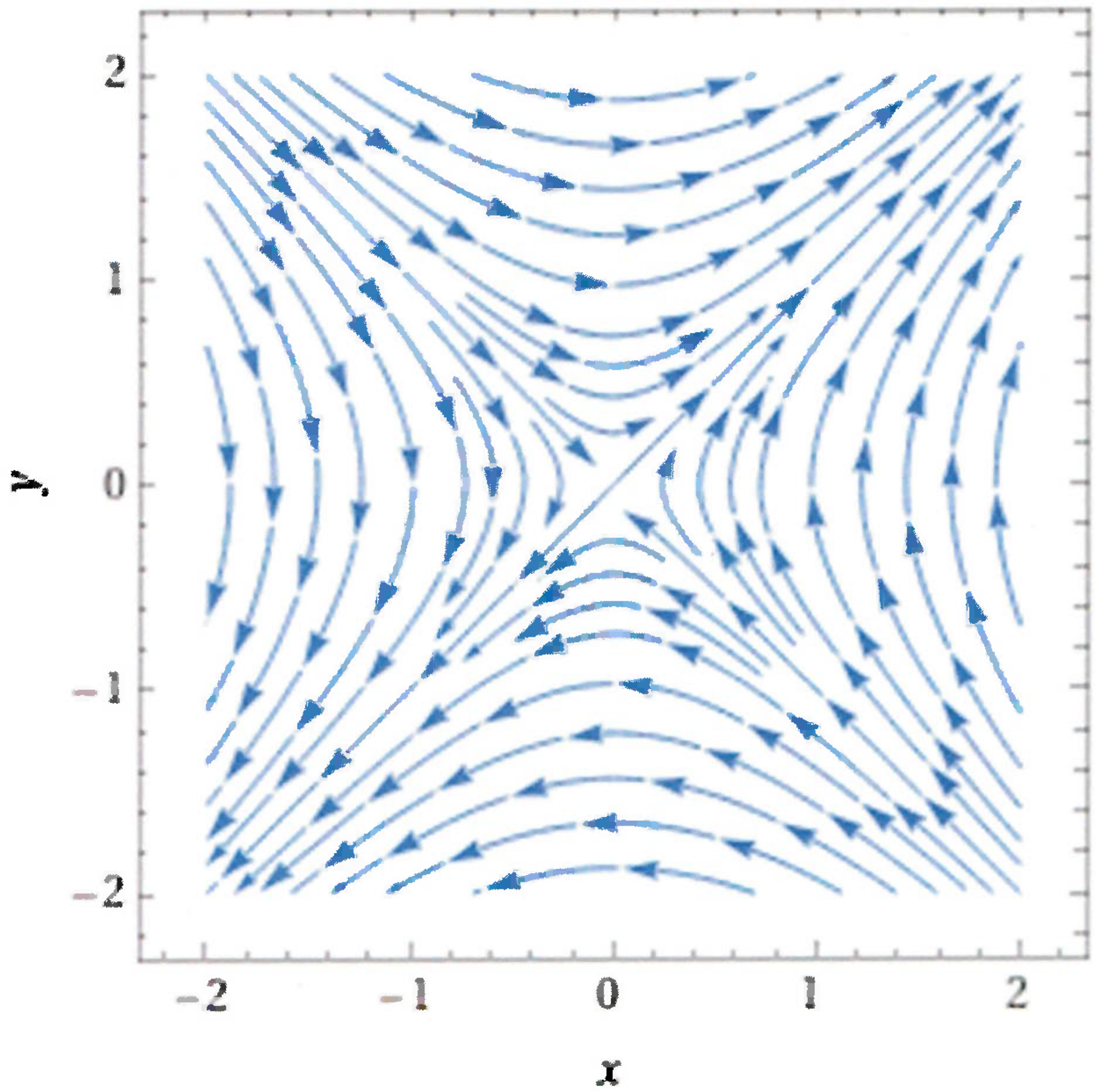
$\lim_{t \to -\infty} \vec{x} \to c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to$ parallel to $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (eigenvector)



solutions approach this $t \to \infty$

solutions approach this $t \to -\infty$

next time: complex eigenvalues



2
1
0
-1
-2
y
-2
-1
0
1
2
x