7.3 The Eigenvalue Method (continued)

$\vec{x}' = A\vec{x}$

Solution: $\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$

$\lambda, \vec{v}$ are the eigenvalue/eigenvector pairs

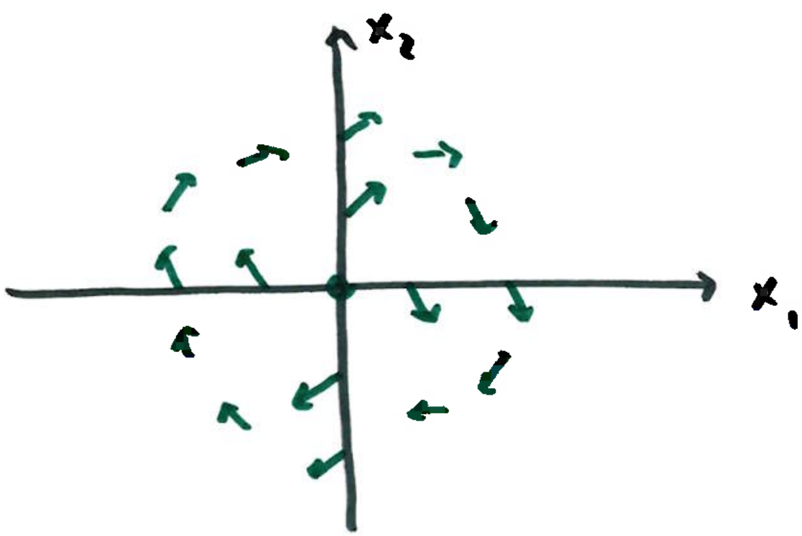
if λ's are real and distinct, then the $\vec{v}$'s are always linear independent so solution is easy to put together

let's now look at complex eigenvalues

<u>example</u>

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\vec{x}$$

let's take a look at the slope field

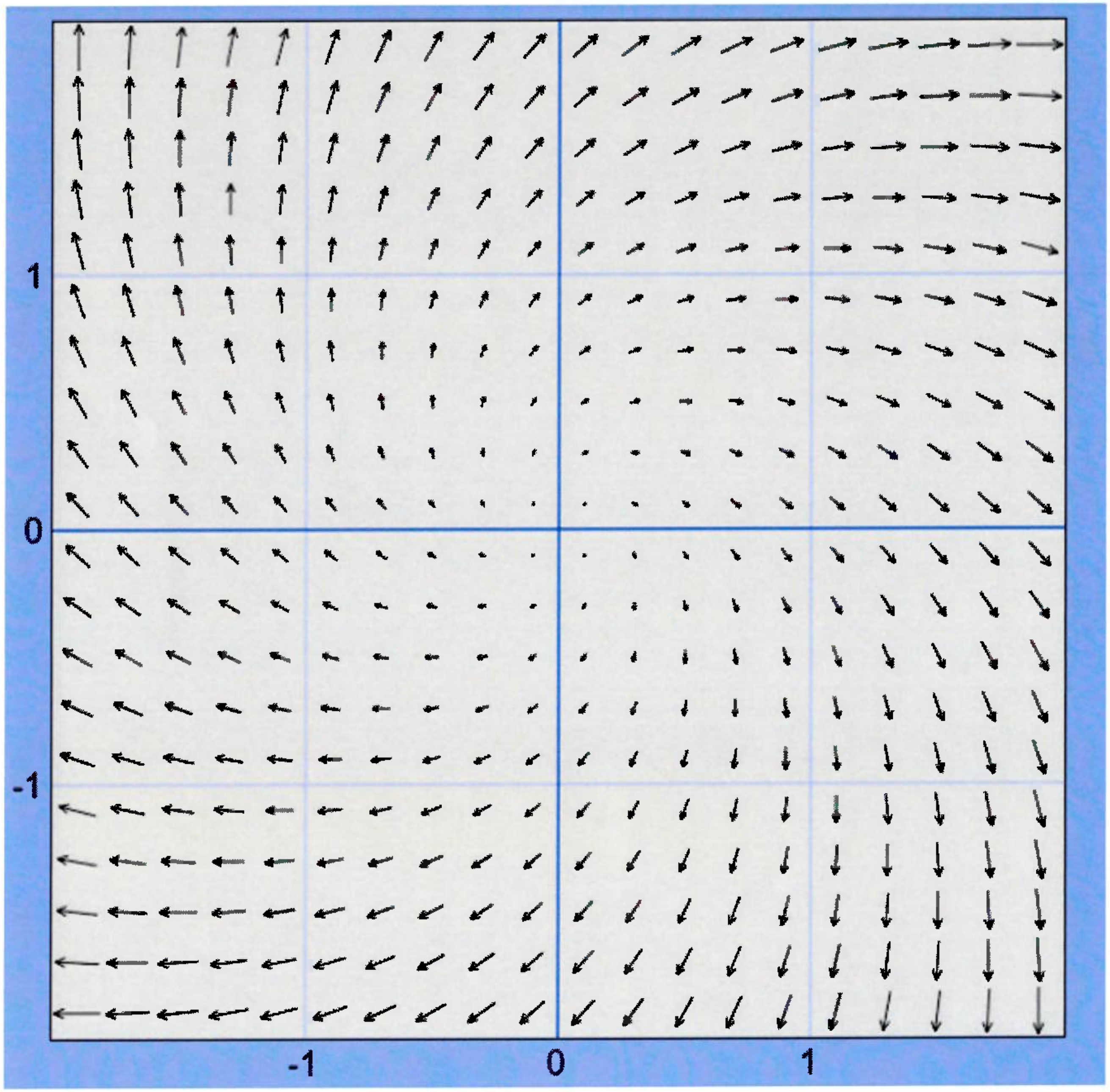


$\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

rotation in the slope field

$\rightarrow$ λ's are complex


1
0
-1
-1
0
1

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$$

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \qquad (1-\lambda)^2 + 1 = 0 \qquad (1-\lambda)^2 = -1$$

$1-\lambda = i$, $1-\lambda = -i$

$\lambda = 1-i$, $\lambda = 1+i$

conjugate pairs

eigenvector : $(A - \lambda I)\vec{v} = \vec{0}$

$\underline{\lambda = 1+i}$

$$\begin{bmatrix} -i & 1 & 0 \\ -1 & -i & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

pick $\vec{v}$ such that $(A - \lambda I)\vec{v} = \vec{0}$

$\underline{\lambda = 1-i}$

$$\begin{bmatrix} i & 1 & 0 \\ -1 & i & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

as expected, $\vec{v}$'s are also conjugate pairs each independent solution is $e^{\lambda t}\vec{v}$

Euler's formula

$$e^{it} = \cos(t) + i\sin(t)$$

$\lambda = 1+i$, $\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$$e^{(1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} = e^t e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} = e^t(\cos t + i\sin t)\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^t\begin{bmatrix} \cos t + i\sin t \\ -\sin t + i\cos t \end{bmatrix} = e^t\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i\,e^t\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$i^2\sin t = -\sin t$

$\lambda = 1-i$, $\vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\cos(-t) = \cos(t)$ $\quad$ $\sin(-t) = -\sin(t)$

$$e^{(1-i)}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^t e^{i(-t)}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^t(\cos t - i\sin t)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= e^t\begin{bmatrix} \cos t - i\sin t \\ -\sin t - i\cos t \end{bmatrix} = e^t\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} - i\,e^t\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

again, complex conjugate pair

general solution: $\vec{x} = c_1 e^{\lambda_1 t}\vec{v_1} + c_2 e^{\lambda_2 t}\vec{v_2}$

$$\vec{x} = c_1\left\{e^t\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix} + ie^t\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}\right\} + c_2\left\{e^t\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix} - ie^t\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}\right\}$$

real-valued function, we want to eliminate i (c_1, c_2 are complex)

$$= \underbrace{(c_1+c_2)}_{\text{"}c_1\text{"}} e^t\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix} + \underbrace{i(c_1-c_2)}_{\text{"}c_2\text{"}} e^t\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$$

$$\boxed{\vec{x} = c_1 \underbrace{e^t\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}}_{\text{real part of either solution}} + c_2 \underbrace{e^t\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}}_{\text{imaginary part of either solution}}}$$

complex λ's: identify real and imaginary parts of either solution

then use them as the basis solutions of the general solution example

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x}$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \qquad \lambda^2 + 1 = 0 \qquad \lambda = \pm i$$

just need one of these solutions

$\lambda = -i$ $\quad (A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} i & 1 & 0 \\ -1 & i & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

how about $\begin{bmatrix} i \\ 1 \end{bmatrix}$?

also ok, will affect the form of solution but still the same

let's use $\lambda = -i$, $\vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$e^{\lambda t}\vec{v} = e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = (\cos t - i \sin t)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= \begin{bmatrix} \cos t - i \sin t \\ -\sin t - i \cos t \end{bmatrix} = \boxed{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}} - i \boxed{\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}$$

use these in gen. solution

what would it look like if we picked $\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$$e^{\lambda t}\vec{v} = (\cos t - i\sin t)\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sin t + i\cos t \\ \cos t - i\sin t \end{bmatrix} = \boxed{\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}} + i\boxed{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}$$

or these

general solution

$$\boxed{\vec{x} = c_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}$$

or

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example

$$\vec{x}' = \begin{bmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{bmatrix}\vec{x}$$

possibilities: 3 real λ's

1 real, 2 complex

$$\lambda = 3,\ -1+i,\ -1-i$$

$$\vec{v} = \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix},\ \begin{bmatrix} -4-i \\ -9+2i \\ 17 \end{bmatrix},\ \begin{bmatrix} -4+i \\ -9-2i \\ 17 \end{bmatrix}$$

$$e^{3t}\begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix}$$

the other two, follow previous examples

→ form one solution, pick out real and imaginary parts to go into the gen. solution