7.6 Multiple Eigenvalue Solutions

$\vec{x}' = A\vec{x}$ $\quad\quad$ $\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$

$\vec{v}$'s are linearly independent

never a problem if λ's are distinct or complex

potential issue if λ's are repeated

$$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\vec{x}$$

$\lambda = 1, 1$ repeated

algebraic multiplicity of two

find eigenvectors

$(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} r \\ t \end{bmatrix} \quad r, t \text{ reals}$$

$$= r\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

eigenspace dimension is two

two basis vectors

geometric multiplicity of two

let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solution $\vec{x} = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

if there are enough eigenvectors to pair up with the eigenvalues we say A is <u>complete</u>. solution is formed as usual.

$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$ $\lambda = 1, 1$

$(A - \lambda I)\vec{v} = \vec{0}$ $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\vec{v} = \begin{bmatrix} r \\ 0 \end{bmatrix} = r \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

eigenvector (one!)
eigenspace dimension is one

if we are missing eigenvectors, we say the matrix A is <u>defective</u>

solution: $\vec{x} = c_1 e^{\lambda t} \vec{v}_1 + c_2 e^{\lambda t} \vec{v}_2$

$= c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \left(\; ? \; \right)$

we've seen similar issue before: $y'' + 10y' + 25y = 0$

$r^2 + 10r + 25 = 0 \rightarrow r = 5, 5$

$y_1 = e^{5t}$ $y_2 = te^{5t}$

So we might guess the second solution is $e^t t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

or $\vec{x}_1 = e^{\lambda t}\vec{v}_1$ and $\underbrace{\cancel{\vec{x}_2 = e^{\lambda t}}\, t\vec{v}_1}_{\text{BUT this does NOT work}}$

we need to form $\vec{x}_2$ this way: $\boxed{\vec{x}_2 = e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)}$

how to find $\vec{v}_2$?

Sub $\vec{x}_2 = e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$ into $\vec{x}' = Ax$

$$\vec{x}_2 = te^{\lambda t}\vec{v}_1 + e^{\lambda t}\vec{v}_2$$

$$\vec{x}_2' = t\lambda e^{\lambda t}\vec{v}_1 + e^{\lambda t}\vec{v}_1 + \lambda e^{\lambda t}\vec{v}_2$$

$$\underline{\underline{t\lambda e^{\lambda t}}}\vec{v}_1 + \underline{\underline{e^{\lambda t}}}\vec{v}_1 + \underline{\underline{\lambda e^{\lambda t}}}\vec{v}_2 = A\underline{\underline{te^{\lambda t}}}\vec{v}_1 + A\underline{\underline{e^{\lambda t}}}\vec{v}_2$$

$te^{\lambda t}$ terms: $\lambda\vec{v}_1 = A\vec{v}_1 \rightarrow (A - \lambda I)\vec{v}_1 = \vec{0}$ $\vec{v}_1$ is eigenvector (not new)

$e^{\lambda t}$ terms: $\vec{v}_1 + \lambda\vec{v}_2 = A\vec{v}_2 \rightarrow \boxed{(A - \lambda I)\vec{v}_2 = \vec{v}_1}$

this can be solved for $\vec{v}_2$

also, notice $(A-\lambda I)(A-\lambda I)\vec{v}_2 = (A-\lambda I)\vec{v}_1$

$$\boxed{(A-\lambda I)^2\vec{v}_2 = \vec{0}}$$

two ways to find $\vec{v}_2$ (generalized eigenvector)

revisit $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\vec{x}$

$\lambda = 1, 1$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

find $\vec{v}_2$: $(A-\lambda I)\vec{v}_2 = \vec{v}_1$

$$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{v}_2 = \begin{bmatrix} r \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

choose $r=0$

first solution: $\vec{x}_1 = e^{\lambda t}\vec{v}_1 = e^t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

second solution $\vec{x}_2 = e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$

$$= e^t\left(t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$\boxed{\text{general: } \vec{x} = c_1 e^t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t\left(t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)}$$

second way to find $\vec{v_2}$: $(A-\lambda I)^2 \vec{v_2} = \vec{0}$

$A-\lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $(A-\lambda I)^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\vec{v_2}$ = anything $\rightarrow$ choose $\vec{v_2}$ such that $(A-\lambda I)\vec{v_2} \neq \vec{0}$

$\rightarrow$ don't choose $\vec{v_1}$ to be $\vec{v_2}$

recall $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so choose $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

then $(A-\lambda I)\vec{v_2} = \vec{v_1}$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as expected

$\boxed{(A-\lambda I)^k = \underset{\text{matrix}}{[0]} \text{ if we are missing } k-1 \text{ eigenvectors}}$

this is the more efficient way to find missing eigenvectors for 3x3 or beyond example

$$\vec{x}' = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 6 & 4 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$$

$\lambda = 2, 2, 6$ $\quad\quad$ $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ for $\lambda = 6$

$\underline{\lambda = 2}$ $\quad (A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -4 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

two free variables → eigenspace dimension <u>two</u>

→ two vectors

... $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

form solution as usual

$$\vec{x} = c_1 e^{6t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

example $\vec{x}\,' = \begin{bmatrix} -11 & 0 & -4 \\ -1 & -9 & -1 \\ 1 & 0 & -7 \end{bmatrix} \vec{x}$

$\lambda = -9, -9, -9$

the only true eigenvector from $(A-\lambda I)\vec{v} = \vec{0}$ is $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

missing two $\rightarrow$ $(A-\lambda I)^{2+1} = [0]$

find $\vec{v}_3$: $(A-\lambda I)^3 \vec{v}_3 = \vec{0}$

then $\vec{v}_2$: $(A-\lambda I)\vec{v}_3 = \vec{v}_2$

then $\vec{v}_1$: $(A-\lambda I)\vec{v}_2 = \vec{v}_1$

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ choose $\vec{v}_3$ arbitrarily EXCEPT or $\vec{0}$

pick $\boxed{\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$

$(A-\lambda I)\vec{v}_3 = \vec{v}_2 = \begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \vec{v}_2}$

$$(A-\lambda I)\vec{v}_2 = \vec{v}_1$$

$$\begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Sometimes matches the true eigenvector but always

solution

$$\vec{x} = c_1 \underbrace{e^{-5t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{e^{\lambda t}\vec{v}_1} + c_2 \underbrace{e^{-5t}\left(t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}\right)}_{e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)}$$

$$+ c_3 \underbrace{e^{-5t}\left(\frac{t^2}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)}_{e^{\lambda t}\left(\frac{t^2}{2}\vec{v}_1 + t\vec{v}_2 + \vec{v}_3\right)}$$