7.6 Multiple Eigenvalue Solutions (part 2)

$\vec{x}' = A\vec{x}$ λ repeated, each one has an eigenvector $\rightarrow$ A is complete

solution formed as usual

$$\vec{x} = c_1 e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}\vec{v}_2 + \cdots + c_n e^{\lambda t}\vec{v}_n$$

if A is defective, then there is not enough eigenvectors

need generalized eigenvectors

last time: 3×3 A, λ repeated 3 times, 1 true eigenvector

find $\vec{v}_3$: $\underbrace{(A-\lambda I)^3}_{[0] \text{ matrix}} \vec{v}_3 = \vec{0}$ # of missing eigenvector plus one

$\vec{v}_3$ is arbitrary, pick ANY $\vec{v}_3$ except the true eigenvector or $\vec{0}$

find $\vec{v}_2$: $(A-\lambda I)\vec{v}_3 = \vec{v}_2$

find $\vec{v}_1$: $(A-\lambda I)\vec{v}_2 = \vec{v}_1$ $\rightarrow$ may or may not be the true eigenvector solutions: $\vec{x}_1 = e^{\lambda t}\vec{v}_1$

$\vec{x}_2 = e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$

$\vec{x}_3 = e^{\lambda t}\left(\frac{t^2}{2}\vec{v}_1 + t\vec{v}_2 + \vec{v}_3\right)$

general solution: $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$

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if A is 3×3, λ repeated 3 times, 2 true eigenvectors

missing one

find $\vec{v}_3$: $\underbrace{(A-\lambda I)^2}_{[0]}\vec{v}_3 = \vec{0}$

$\vec{v}_3$ = anything except $\vec{0}$ except true eigenvectors

find $\vec{v}_2$: $(A-\lambda I)\vec{v}_3 = \vec{v}_2$

find $\vec{v}_1$: either of the two true eigenvectors

solutions: $\vec{x}_1 = e^{\lambda t}\vec{v}_1$

$\vec{x}_2 = e^{\lambda t}\vec{v}_2$

$\vec{x}_3 = e^{\lambda t}(t\vec{v}_2 + \vec{v}_3)$

example $\vec{x}\,' = \begin{bmatrix} -10 & -9 & 0 \\ 1 & -4 & 0 \\ 1 & 3 & -7 \end{bmatrix} \vec{x}$

$\lambda = -7, -7, -7$

true eigenvectors: $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

missing one: $(A - \lambda I)^2 = [0]$

$$A - \lambda I = \begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

$$(A-\lambda I)^2 = \begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

find $\vec{v}_3$: $(A-\lambda I)^2 \vec{v}_3 = \vec{0}$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

not $\vec{0}$, not linear combo of the two true eigenvectors

↖ usually a good choice for the arbitrary gen. eigenvector find $\vec{v}_2$: $(A-\lambda I)\vec{v}_3 = \vec{v}_2$

$$\begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

find $\vec{v}_1$: pick either of the two true eigenvectors

$$\vec{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

solutions: $\vec{x}_1 = e^{\lambda t}\vec{v}_1 = e^{-7t}\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

$\vec{x}_2 = e^{\lambda t}\vec{v}_2 = e^{-7t}\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

$\vec{x}_3 = e^{\lambda t}(t\vec{v}_2 + \vec{v}_3) = e^{-7t}\left(t\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

general solution: $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$

$$A = \begin{bmatrix} a_{11} & & \\ a_{21} & & \\ & & \end{bmatrix} \qquad B = \left[\begin{matrix} b_{11} & & \\ & & \\ & & \end{matrix}\right.$$

$\det(A) = 5 \qquad \det(B) = 6$

find $\det(C)$ where

$$C = \underbrace{\begin{bmatrix} a_{11} & 2a_{12} & a_{13} \\ a_{21} & 2a_{22} & \\ & & \end{bmatrix}}_{\text{A w/ col 2 mult. by 2}} \underbrace{\begin{bmatrix} 3b_{11} & b_{12}+2b_{11} & b_{13} \\ 3b_{21} & b_{22}+2b_{21} & b_{23} \\ & b_{32}+2b_{31} & \end{bmatrix}}_{\text{B w/ col 1 mult. by 3 and col 2 added to by twice of col 1}}$$

multiplies det(A) by 2

mult. det(B) by 3

does NOT change det(B)

if $C = AB$

then $\det(C) = \det(A)\det(B)$

here, $\det(C) = \underbrace{\det(\quad)}_{2\det(A)}\underbrace{\det(\quad)}_{3\det(B)}$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & 1 \\ 2 & -4 & 0 \end{bmatrix}$$ find A^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 & 1 & 0 \\ 2 & -4 & 0 & 0 & 0 & 1 \end{array}\right]$$

do row ops
turn this into
identity

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 6 & 2 & 1 & 1 & 0 \\ 0 & -8 & -2 & -2 & 0 & 1 \end{bmatrix}$$

$\frac{8}{3} - \frac{6}{3}$

$\frac{8}{6} - \frac{12}{6}$

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & 1/6 & 1/6 & 0 \\ 0 & 0 & 2/3 & -2/3 & 4/3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & -1/6 & & \\ 0 & 0 & 2 & -2 & 4 & 3 \end{bmatrix}$$

$-2 \cdot -\frac{1}{6} + \frac{1}{6} =$

$4 \cdot$

you get the idea ...

A is $m \times n$ $\quad \vec{b}$ is $m \times 1$ $\qquad$ m rows and n columns

$A\vec{x} = \vec{b}$ has inf. solutions

i) $m \leq n$ → fewer rows than cols

ii) $n \leq m$ → fewer cols than rows

iii) rank $A = n$

iv) rank $A < n$ → yes

v) $\det A = 0$ → zero row(s)
→ free vars → inf. sols

\# pivots = cols → no inf. sols

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 5 & \boxed{6} & 7 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{b}$$

max pivots = 2

4 variables

2 free → inf. solutions

$$\begin{bmatrix} \boxed{1} & 4 \\ 2 & \boxed{5} \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

max pivots 2

2 variables

possible to have inf. solutions