7.6 Multiple Eigenvalue Solutions (part 3)

$\vec{x}' = A\vec{x}$ A defective: not enough eigenvectors

generalized eigenvectors: $(A-\lambda I)^k \vec{v}_i = \vec{0}$ k: # missing + 1

$(A-\lambda I)\vec{v}_i = \vec{v}_{i-1}$

build k of these

supplement with true eigenvectors to form a complete set of eigenvectors.

<u>example</u>

$$\vec{x}' = \begin{bmatrix} 3 & -10 & 0 & -5 \\ 0 & 3 & 0 & 0 \\ 15 & -30 & -2 & -15 \\ 0 & -10 & 0 & -2 \end{bmatrix} \vec{x}$$

$\lambda = 3, 3, -2, -2$

if $\underline{\lambda = 3}$ $(A-\lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cccc|c} 0 & -10 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 15 & -30 & -5 & -15 & 0 \\ 0 & -10 & 0 & -5 & 0 \end{array}\right]$$

$$\to \begin{bmatrix} \boxed{1} & 0 & -1/3 & 0 & 0 \\ 0 & \boxed{1} & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \underbrace{0} & \underbrace{0} & 0 \end{bmatrix}$$

free → eigenspace dimension two

so two true eigenvectors, no need for generalized ones

$$\ldots \quad \vec{v} = \cancel{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

$\underline{\lambda = 2}$ $\quad (A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 1 & -10 & 0 & -5 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 15 & -30 & -4 & -15 & 0 \\ 0 & -10 & 0 & 4 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} \boxed{1} & 0 & 0 & -1 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \underbrace{0} & \underbrace{0} & 0 \end{bmatrix} \quad \ldots \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

free → dimension two

A is complete, solution is formed the normal way example

$$\vec{x}\,' = \begin{bmatrix} 1 & 3 & 7 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -6 & -14 & 1 \end{bmatrix} \vec{x}$$

$\lambda = 1, 1, 1, 1$

$\underline{\lambda = 1}$ $\quad (A - \lambda I)\vec{v} = \vec{0}$

pivots

$$\left[\begin{array}{cccc|c} 0 & 3 & 7 & 0 & 0 \\ 0 & -2 & -4 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & -6 & -14 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free → dimension Two

true eigenvectors: 2nd, 3rd elements zero

1st, 4th free

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

missing two so $(A - \lambda I)^3 \vec{v} = [0]$

build 3 generalized eigenvectors

$\vec{v}_4$: $(A-\lambda I)^3 \vec{v}_4 = \vec{0}$

pick $\vec{v}_4 \neq \vec{0}$, not true eigenvector

$$\vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\vec{v}_3$: $(A-\lambda I)\vec{v}_4 = \vec{v}_3$

$$\begin{bmatrix} 0 & 3 & 7 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -6 & -14 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -6 \end{bmatrix}$$

$\vec{v}_2$: $(A-\lambda I)\vec{v}_3 = \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$

got 3 of these, $(A-\lambda I)^3 = [0]$ tells us we need 3, so stop

$\vec{v}_1$: either of the true eigenvectors $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

general solution

$$\vec{x} = c_1 e^{t}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + c_2 e^{t}\begin{bmatrix}1\\0\\0\\-2\end{bmatrix} + c_3 e^{t}\left(t\begin{bmatrix}1\\0\\0\\-2\end{bmatrix} + \begin{bmatrix}3\\-2\\1\\-6\end{bmatrix}\right)$$
$$+ c_4 e^{t}\left(\frac{t^2}{2}\begin{bmatrix}1\\0\\0\\-2\end{bmatrix} + t\begin{bmatrix}3\\-2\\1\\-6\end{bmatrix} + \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)$$

<u>example</u>

$$\vec{x}\,' = \begin{bmatrix}2 & 1 & 0 & 1\\0 & 2 & 1 & 0\\0 & 0 & 2 & 1\\0 & 0 & 0 & 2\end{bmatrix}\vec{x}$$

triangular matrix → λ's are diagonal elements

$\lambda = 2, 2, 2, 2$

$(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix}0 & 1 & 0 & 1 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

x_1 free

$x_4 = 0$

$x_3 = 0$

$x_2 + x_4 = 0 \rightarrow x_2 = 0$

$\vec{v} = \begin{bmatrix}r\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$

3 pivots → 4 columns

→ $4 - 3 = 1$ is eigenspace dimension

missing 3 (defect of 3)

$\vec{v_4}$: $(A-\lambda I)^4 \vec{v_4} = \vec{0}$

$\vec{v_4} \neq 0$, not true vector, and not one such that $(A-\lambda I)\vec{v_4} = \vec{0}$

if $\vec{v_4} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $(A-\lambda I)\vec{v_4} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ true one, can't be because $(A-\lambda I)\vec{v} = \vec{0}$ so can't find $\vec{v_3}$

so can't be $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

try $\vec{v_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$(A-\lambda I)\vec{v_4} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ this is not a true eigenvector or $\vec{0}$, so ok.

so, $\vec{v_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{v_3}$: $(A-\lambda I)\vec{v_4} = \vec{v_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{v_2}$: $(A-\lambda I)\vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ true

problem: $\vec{v_1} = (A-\lambda I)\vec{v_2} = \vec{0}$

so, $\vec{v_3}$ is not right $\rightarrow$ $\vec{v_4}$ not right

after a few more tries, $\vec{v_4} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

then follow steps.

usually we don't do systems beyond 3×3 by hand