7.4 Solution Curves of Linear Systems (part 1)

the system $\vec{x}' = \begin{bmatrix} 4 & 7 \\ 7 & 4 \end{bmatrix} \vec{x}$ has eigenvalues / eigenvectors

$$\lambda = -3, \ 11$$

$$\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution: $\vec{x} = c_1 e^{-3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{11t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$x_1 = -c_1 e^{-3t} + c_2 e^{11t}$$

$$x_2 = c_1 e^{-3t} + c_2 e^{11t}$$

if we plot x_1 vs x_2 → relationship between them

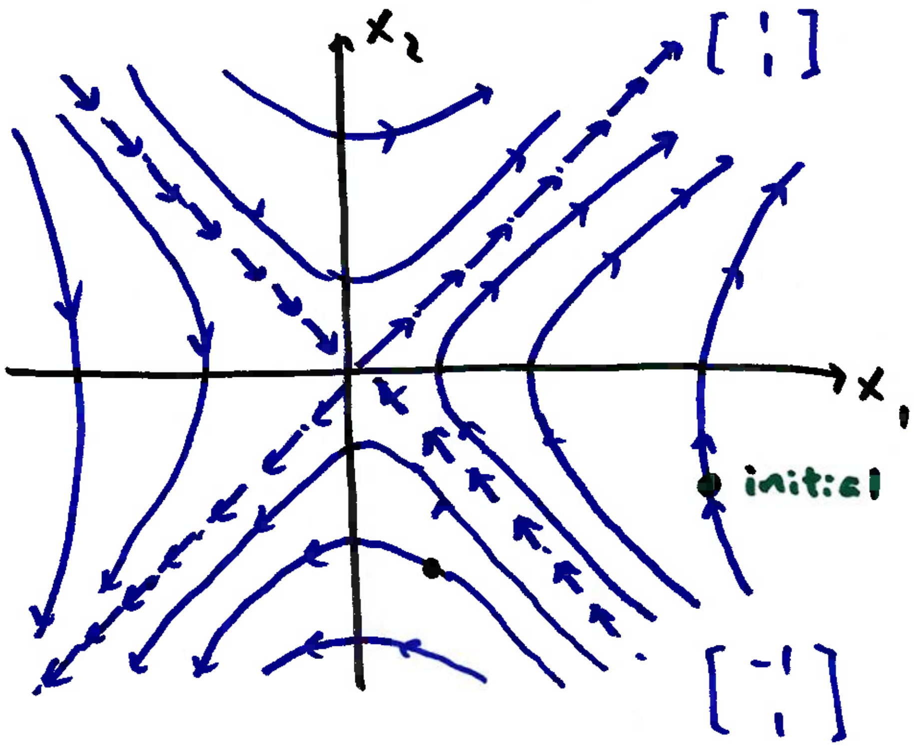
→ <u>phase portrait</u>

provides qualitative understanding of solutions regardless of initial conditions

$$\vec{x} = \underbrace{c_1 e^{-3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{dominates when } t \to -\infty} + \underbrace{c_2 e^{11t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{dominate if } t \text{ is large } (t \to \infty)} \qquad \cancel{t = 0 : \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

so, as $t \to \infty$, all solutions approach the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

as $t \to -\infty$, all " " " " $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ } behave like asymptotes



if we start along $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $(c_1 = 0)$

$x_1 \to \infty$, $x_2 \to \infty$ → $\lambda > 0$

→ away from origin

in Q III, go to $-\infty$

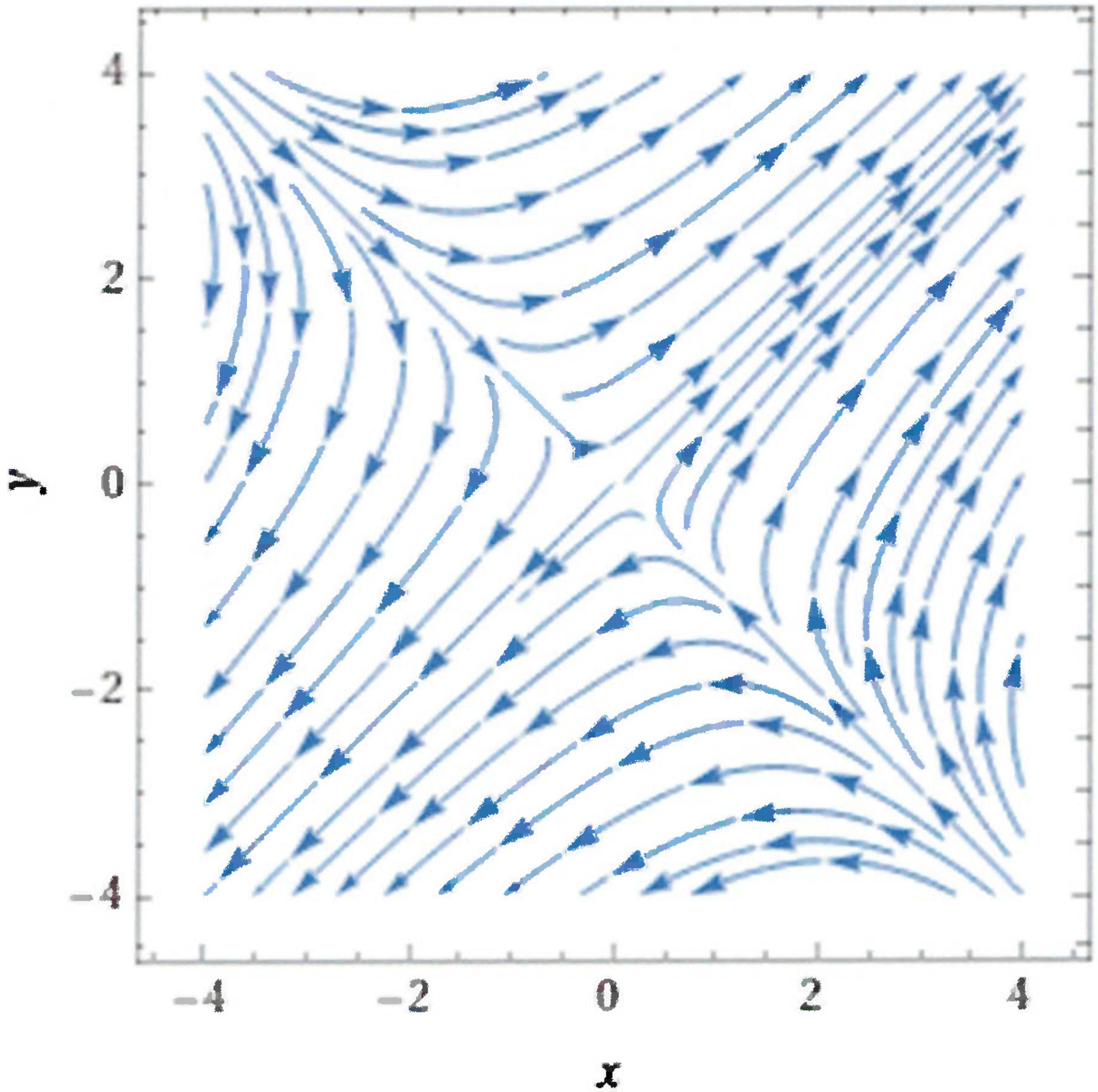
along $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $e^{-3t} \to 0$ as $t \to \infty$

so solutions approach origin → $\lambda < 0$

other curves follow the asymptotes

in this case, the origin is a saddle point → some approach some go away from origin

→ λ's are real and of opposite signs

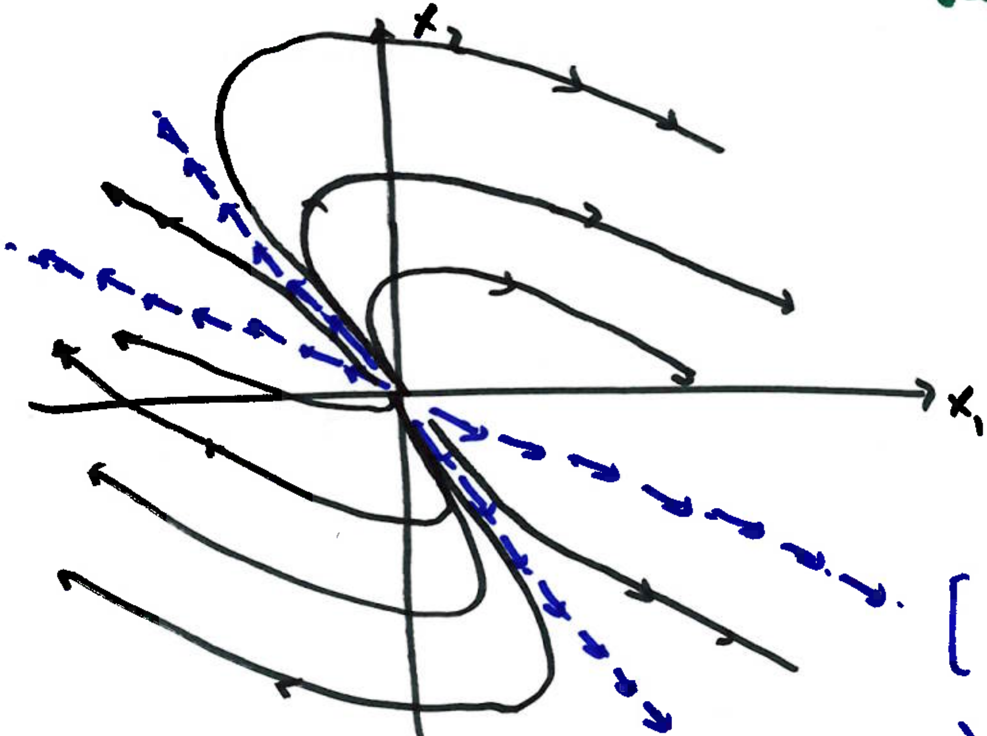


4
2
0
−2
−4
y
−4
−2
0
2
4
x

example $\vec{x}\,' = \begin{bmatrix} 6 & 5 \\ -3 & -2 \end{bmatrix} \vec{x}$

$\lambda = 3,\ 1$

$\vec{v} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\Bigg\}$ $\vec{x} = \underbrace{c_1 e^{3t} \begin{bmatrix} -5 \\ 3 \end{bmatrix}}_{\text{dominant } t \to \infty} + \underbrace{c_2 e^{t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{dominates before } t \to \infty}$



$\begin{bmatrix} -5 \\ 3 \end{bmatrix}$

$\lambda = 3 > 0$

away from origin

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 1 > 0$

away from origin

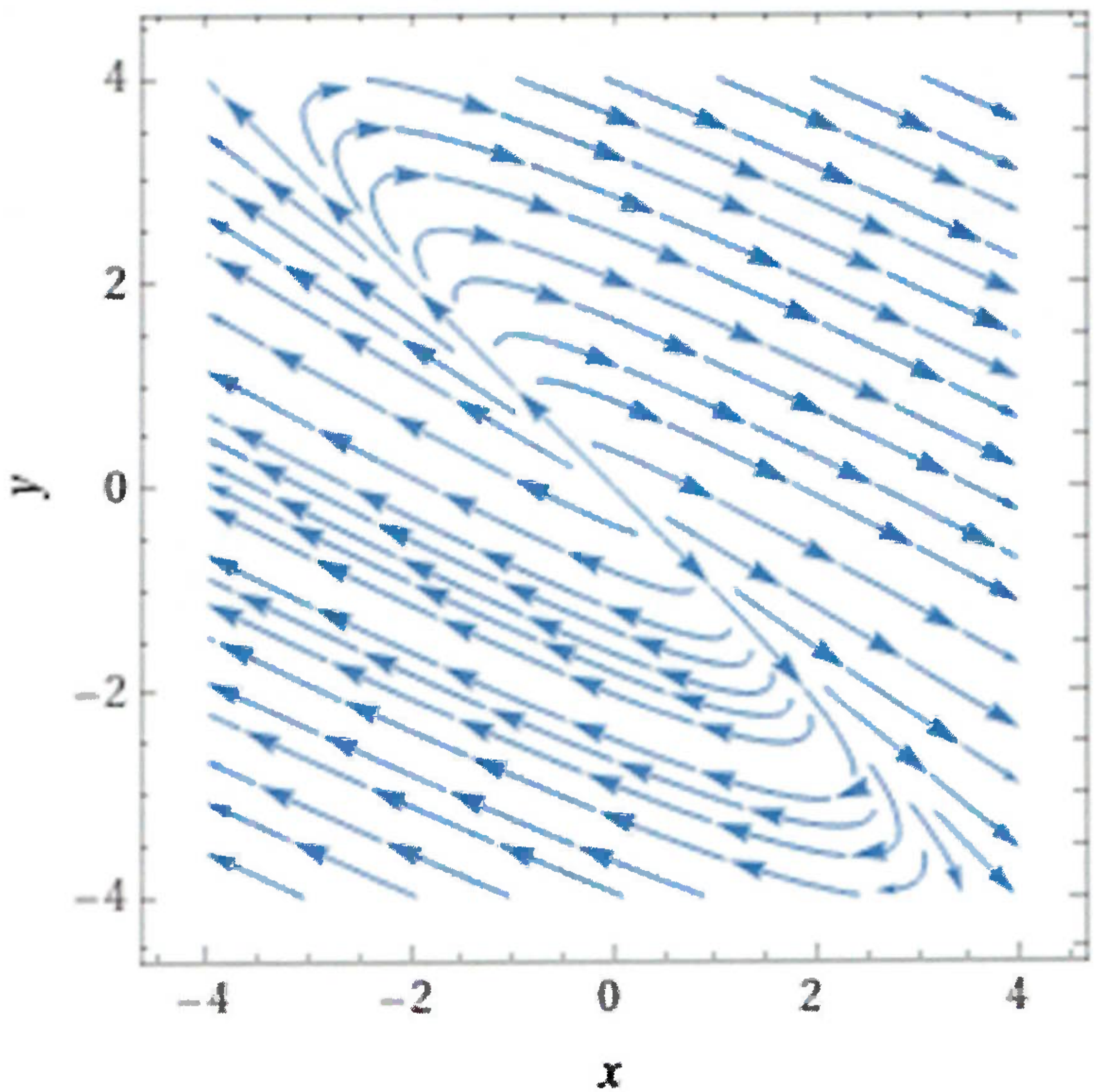
initially follows $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, then as $t \to \infty$, follows $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$

the origin is called is <u>source</u> because all solutions go away from it

it is also an <u>improper node</u> (things follow an asymptote near origin)

origin here: <u>improper nodal source</u>

↳ λ's distinct both positive



4
2
0
−2
−4
y
−4
−2
0
2
4
x

if both λ's are negative and distinct, same general picture w/ arrows reversed. Origin is an improper nodal sink.

example $\vec{x}' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \vec{x}$

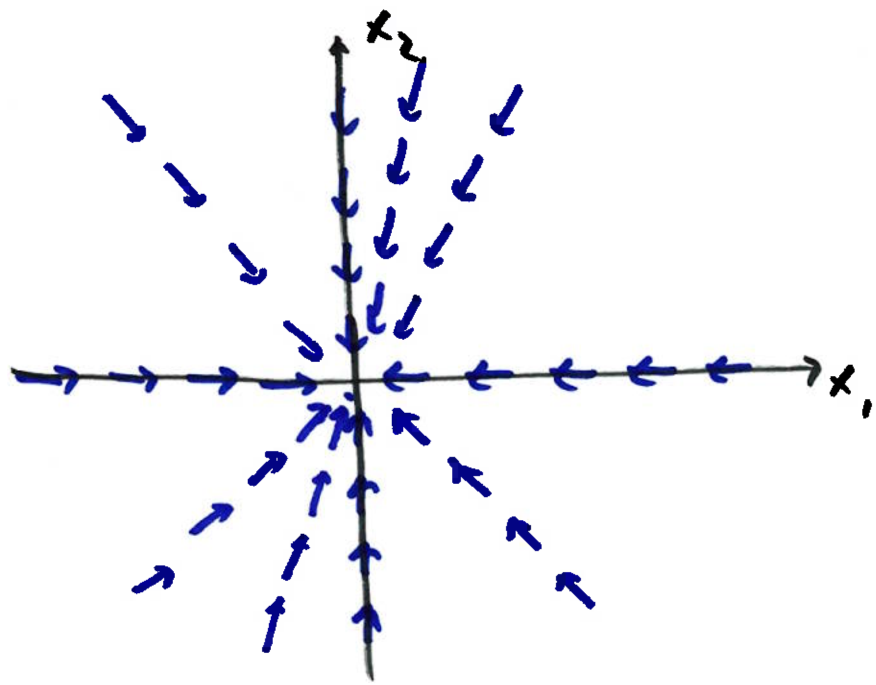
$\lambda = -1, -1$

$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x_1 = c_1 e^{-t} \quad x_2 = c_2 e^{-t}$

$\frac{x_2}{x_1} = \frac{c_2}{c_1} = m \rightarrow \underbrace{x_2 = m x_1}_{\text{lines through origin, slope } m}$



proper nodal sink

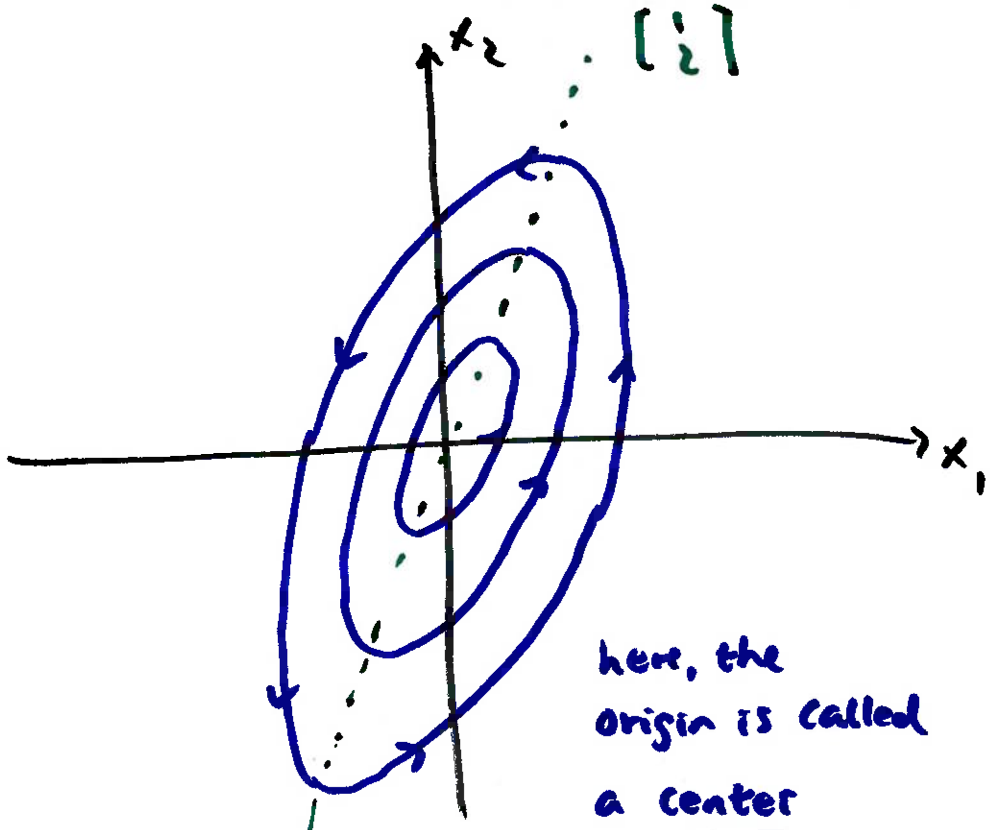
proper because all solutions follow their own curves toward or away from origin and NOT along an asymptote

proper: λ's are repeated AND w/ enough eigenvectors example

$$\vec{x}' = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \vec{x}$$

$$\left.\begin{array}{l} \lambda = i, -i \\ \vec{v} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}, \begin{bmatrix} 1-i \\ 2 \end{bmatrix} \end{array}\right\} \quad \vec{x} = c_1 \begin{bmatrix} \sin t + \cos t \\ 2\sin t \end{bmatrix} + c_2 \begin{bmatrix} -\sin t \\ \cos t - \sin t \end{bmatrix}$$

both x_1 and x_2 are periodic $\rightarrow$ x_2 vs x_1 solutions are ovals (ellipses)



here, the origin is called a center

real part of either eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\rightarrow$ major axis of ellipse

direction?

pick a convenient location, look at direction from $\vec{x}' = A\vec{x}$

here, pick $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\vec{x}' = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

right and up

ccw

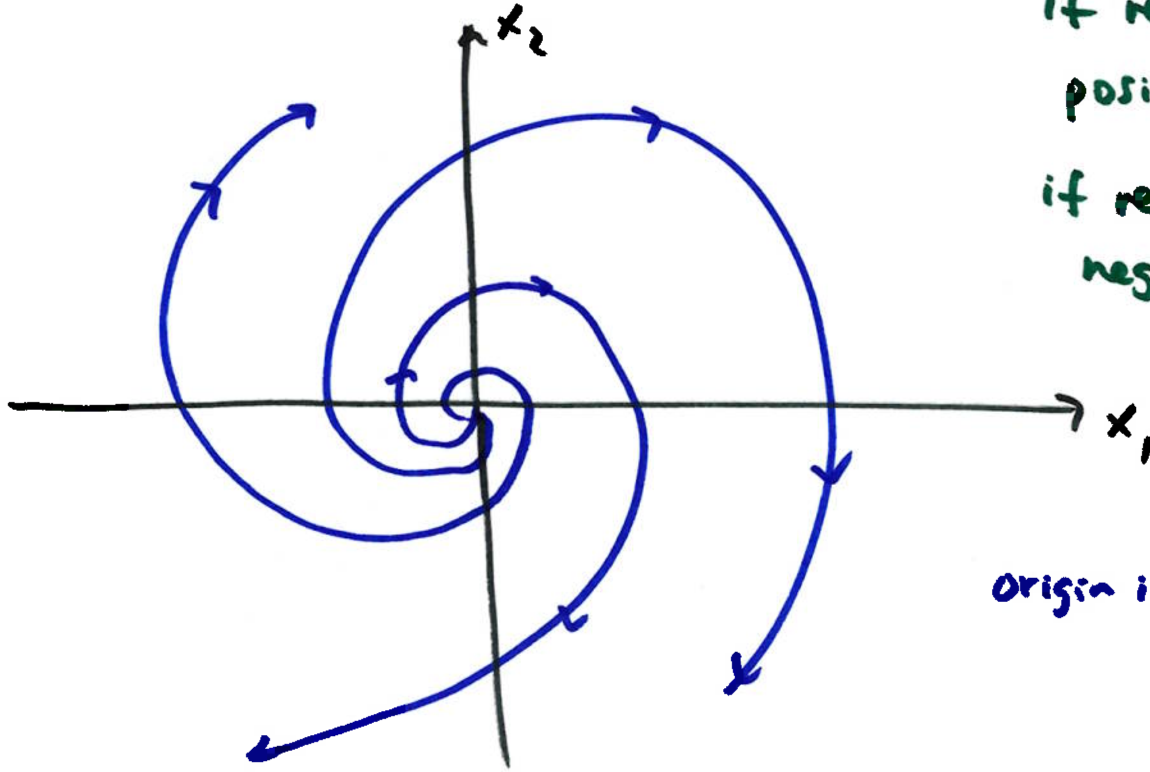
<u>example</u> $\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$

$\left.\begin{array}{l} \lambda = 1-i,\ 1+i \\ \vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix} \end{array}\right\} \vec{x} = c_1 e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$

use the real part of $\vec{v}$ to orient ovals

the presence of e^t means the ovals grows as $t \to \infty$

so ovals $\to$ spirals



if real part of λ is positive $\to$ spiral <u>out</u>

if real part of λ is negative $\to$ spiral <u>in</u>

origin is a <u>spiral source</u> (b/c real part of $\lambda > 0$)

(spiral sink if real part of λ is negative)