1.4 Separable Equations

recall $\frac{dy}{dx} = 2x$ is easy to solve because no y on right side

$$y = \int 2x\,dx = x^2 + c$$

$$\frac{dy}{dx} = 2x \longrightarrow dy = 2x\,dx$$

integrate both sides $\int dy = \int 2x\,dx$

$$y = x^2 + c$$

separable eqs → we can always cleanly separate x and y
integrate both sides to solve

$\frac{dy}{dx} = x^2 + y^2$ is not separable
because we cannot write as $f(y)\,dy = g(x)\,dx$

<u>example</u> $\frac{dy}{dx} - xy = 0$

rewrite : $\frac{dy}{dx} = xy$ separate by dividing or multiplying

$$\frac{1}{y}\,dy = x\,dx$$

integrate both sides

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\boxed{\ln|y| = \tfrac{1}{2}x^2 + c}$$ the implicit form of solution

often we can solve for y explicitly

recall: $\ln a = b \longleftrightarrow a = e^b$

$\ln|y| = \frac{1}{2}x^2 + c$ constant

becomes $|y| = e^{\frac{1}{2}x^2 + c}$

recall: $e^{a+b} = e^a \cdot e^b$

$|y| = e^{\frac{1}{2}x^2 + c}$

$= e^{\frac{1}{2}x^2} \cdot \boxed{e^c}$ constant constant constant

e^c is constant

call it C

$|y| = C e^{\frac{1}{2}x^2}$

often easy to find because we are given $y(x_0) = y_0$

drop $|\ |$ because C can be of any sign

$$\boxed{y = C e^{\frac{1}{2}x^2}}$$

explicit form of solution

example $(1+x)y' = y$

$$\frac{dy}{dx} = y' = \frac{y}{1+x}$$

$$\frac{1}{y}\,dy = \frac{1}{1+x}\,dx$$ separated

$$\int \frac{1}{y}\,dy = \int \frac{1}{1+x}\,dx$$

$$\ln|y| = \ln|1+x| + c$$

exponentiate:

$$|y| = e^{\ln|1+x| + c}$$

$$= e^{\ln|1+x|} \cdot e^{c}$$ recall $e^{\ln a} = a$

$$= |1+x| \cdot C$$

solution: $\boxed{y = C(1+x)}$

example $y' = xy^2 - y^2 \qquad y(1) = 2$

$$\frac{dy}{dx} = y^2(x-1)$$

$$\frac{1}{y^2}\,dy = \cancel{\frac{y}{x-1}\,dx} \quad (x-1)\,dx$$

$$\int \frac{1}{y^2}\,dy = \cancel{\int \frac{1}{x+1}\,dx} \quad \int (x-1)\,dx$$

$$-\frac{1}{y} = \frac{1}{2}x^2 - x + C$$

we can find C now (or wait until after explicit solution)

$$y = \frac{-1}{\frac{1}{2}x^2 - x + C}$$

$$2 = \frac{-1}{-\frac{1}{2} + C}$$

$$-\frac{1}{2} + C = -\frac{1}{2} \qquad C = 0$$

so, $\boxed{y = \frac{-1}{\frac{1}{2}x^2 - x}}$

Lots of applications involve separable equations

constant of proportionality

population : $\frac{dP}{dt} = kP$

compound interest : $\frac{dP}{dt} = kP$

radioactive decay : $\frac{dP}{dt} = kP$

negative

k : decay constant (^{235}U : $k = -3.12 \times 10^{-17}$
uranium 235)

Newton's Law of Cooling :

temp

surrounding

$$\frac{dT}{dt} = k(T - M)$$

time

let's solve $\frac{dT}{dt} = k(T-M)$ $\quad M$: constant

$$\int \frac{1}{T-M}\,dT = \int k\,dt$$

$$\ln|T-M| = kt + c$$

$$T-M = e^{kt+c} = e^{kt}\cdot e^{c} = Ce^{kt}$$

So, $T(t) = M + Ce^{kt}$

in the case of hot coffee in room temp.

$k < 0$

$$\lim_{t\to\infty} T(t) = \lim_{t\to\infty} M + \underbrace{Ce^{kt}}_{\to 0}$$

$k<0$

$$= M \quad \text{(room temp.)}$$

back to $\dfrac{dy}{dx} = xy$

$\dfrac{1}{y}\,dy = x\,dx$ we divided by y → assumed $y \neq 0$

does $y=0$ satisfy the DE?

$y=0 \rightarrow \dfrac{dy}{dx} = 0$ (left)

$\underset{\nwarrow 0}{x y} = 0$ (right)

so, $y=0$ is a solution

("singular solution")

$$\downarrow$$

$y = Ce^{\frac{1}{2}x^2}$

general solution

is $y=0$ contained in this family of solutions?

yes, because we can have $C=0 \rightarrow y = 0 \cdot e^{\frac{1}{2}x^2} = 0$

so we didn't lose any solution even though we divided by y earlier

this is NOT always true, though.

Sometimes the singular solution is NOT parts of the general solution, so is "lost"

→ depends on "linearity" of DE.