1.5 Linear First-Order Equations

linear first-order : $\dfrac{dy}{dx} + \underline{\underline{P(x)}}\,y = \underline{\underline{Q(x)}}$

cannot contain y

$y' + xy = \cos x$ is linear

$y' - \underbrace{y^2}_{y \cdot y} = 1$ is NOT linear

coefficient of y term contains y

Some eqs are separable and linear : $y' = xy \to \frac{1}{y}\,dy = x\,dx$

$y' - xy = 0$

Some separables are not linear : $y' = xy^2$ $\quad \frac{1}{y^2}\,dy = x\,dx$

$y' - (xy)y = 0$ not linear

linear but not separable: $y' = xy + 2 \qquad y' - xy = 2$ linear

as $f(y)\,dy = g(x)\,dx$?

How to solve $\frac{dy}{dx} + P(x)\,y = Q(x)$?

simple example: $\frac{dy}{dx} + \frac{1}{x}y = 0$

notice if we multiply by x

$$x\left(\frac{dy}{dx} + \frac{1}{x}y\right) = x(0)$$

$$\underbrace{x\frac{dy}{dx} + y}_{\text{this is } \frac{d}{dx}(xy) \text{ by product rule}} = 0$$

$$\frac{d}{dx}(xy) = 0$$

so $xy = C$ (deriv. of constant is 0)

therefore, $\boxed{y = \frac{C}{x}}$

ALL first-order linear, we can always multiply by an <u>integrating factor</u> (x is the example above) to turn left side into deriv. of product

First-order eq: $y' + P(x)y = Q(x)$

the integrating factor is $\boxed{I = e^{\int P(x)\,dx}}$

previous example: $y' + \frac{1}{x}y = 0$ $P(x) = \frac{1}{x}$ $Q(x) = 0$

$$I = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x + C}$$

we only need one integrating factor, so choose C to make I convenient (usually $C = 0$)

$$I = e^{\ln x} = x$$

<u>example</u> $xy' + 3y = x \qquad y(1) = 2$

rewrite: $y' + \underbrace{\boxed{\frac{3}{x}}}_{P(x)} y = 1$ make coefficient of y' a 1

integrating factor: $I = e^{\int P(x)\,dx} = e^{\int \frac{3}{x}\,dx}$

$= e^{3\ln x + C}$ choose this $C = 0$

$= e^{3\ln x} = e^{\ln x^3} = x^3$

multiply $y' + \frac{3}{x}y = 1$ by x^3 on both sides

$$\underbrace{x^3 y' + 3x^2 y}_{\frac{d}{dx}(x^3 y)} = x^3$$

I

$$\frac{d}{dx}(x^3y) = x^3$$

$$x^3y = \frac{1}{4}x^4 + C$$ this C depends on initial condition

$$y = \frac{1}{4}x + \frac{C}{x^3}$$ $y(1)=2$

$$2 = \frac{1}{4} + C \qquad C = \frac{7}{4}$$

so, $\boxed{y = \frac{1}{4}x + \frac{7}{4x^3}}$

<u>example</u> : $y' - 4y = e^{4x} \qquad y(0)=0$

already standard form

$P(x) = -4$

$$I = e^{\int -4\,dx} = e^{-4x+C}$$

C doesn't matter, let's try $C=1$

$I = e^{-4x+1} = e^1 \cdot e^{-4x}$

multiply $y' - 4y = e^{4x}$ by $e \cdot e^{-4x}$

$\cancel{e} \cdot e^{-4x}(y' - 4y) = \cancel{e} \cdot e^{-4x}(e^{4x})$

$e^{-4x}y' - 4e^{-4x}y = 1$

if we found I correctly, left is always $\frac{d}{dx}(Iy)$

$\frac{d}{dx}(e^{-4x}y) = 1$

verify!

integrate: $e^{-4x}y = x + c$

$y = xe^{4x} + ce^{4x}$ $\quad y(0) = 0$

$0 = c$

$\boxed{y = xe^{4x}}$

example Free fall and terminal velocity

$$\frac{dv}{dt} = 10 - 2v \qquad v(0) = 0$$

$\uparrow$ $d = 2v$ (twice of velocity)

$\downarrow$ $g = 10$ (gravity)

terminal velocity : $\lim_{t\to\infty} v(t)$

$$\frac{dv}{dt} + 2v = 10 \qquad P(t) = 2 \quad I = e^{\int 2\,dt} = e^{2t}$$

$$e^{2t}\frac{dv}{dt} + 2e^{2t}v = 10e^{2t}$$

$$\frac{d}{dt}\left(e^{2t}v\right) = 10e^{2t}$$

verify

$$e^{2t}v = \int 10e^{2t}\,dt = 5e^{2t} + C$$

$$v(t) = 5 + Ce^{-2t} \qquad v(0) = 0$$

$$0 = 5 + C \qquad C = -5$$

$$\boxed{v(t) = 5 - 5e^{-2t}} \qquad \boxed{\lim_{t\to\infty} v(t) = 5} \text{ terminal}$$