1.5 First-order Linear Equations (continued)

$y' + P(x)y = Q(x) \qquad y(x_0) = y_0$

integrating factor $I = e^{\int P(x)\,dx}$

then multiply by I on both sides

solve $\frac{d}{dx}(Iy) = IQ$

today: application (mixing problem)

<u>example</u> A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Water containing $\frac{1}{2}$ lb of salt per gal is poured in at 4 gal/min. The well mixed solution is let out of the tank at 4 gal/min. Find function describing amount of salt in the tank as a function of time.

set up differential eq $\rightarrow$ how amount of salt changes

$y(t)$: amount of salt in tank (lb)

$\frac{dy}{dt}$: rate of change of amount $= (\text{rate in}) - (\text{rate out})$

$\frac{dy}{dt} = (\text{concentration in})(\text{flow rate in}) - (\text{conc out})(\text{flow out})$

salt in tank

$= \left(\frac{1}{2}\ \text{lb/gal}\right)\left(4\ \text{gal/min}\right) - \left(\frac{y}{600}\ \text{lb/gal}\right)\left(4\ \text{gal/min}\right)$

volume of water in tank

$\boxed{y' = 2 - \frac{1}{150} y}$ DE describing y $\quad y(0) = 40$

separable? yes $\quad f(y)\,dy = g(x)\,dx \quad y' = \frac{300-y}{150} \rightarrow \frac{1}{300-y}\,dy = \frac{1}{150}\,dt$

linear? yes

$y' + \frac{1}{150} y = 2$

integrating factor: $I = e^{\int \frac{1}{150} dt} = e^{\frac{1}{150}t}$

$$e^{\frac{1}{150}t}\left(y' + \frac{1}{150}y\right) = e^{\frac{1}{150}t} \cdot 2$$

$$e^{\frac{1}{150}t}y' + \frac{1}{150}e^{\frac{1}{150}t}y = 2e^{\frac{1}{150}t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{150}t}y\right) = 2e^{\frac{1}{150}t}$$

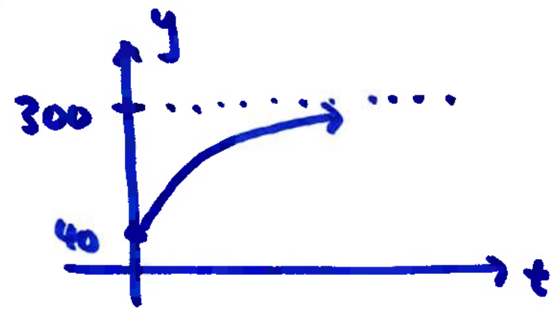
$$e^{\frac{1}{150}t}y = 300e^{\frac{1}{150}t} + C$$

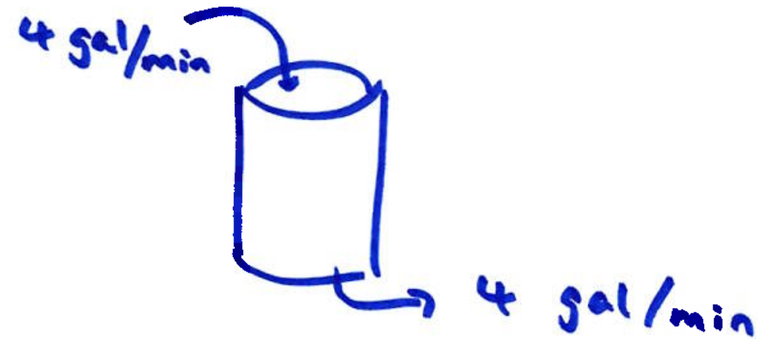
$$y = 300 + Ce^{-\frac{1}{150}t} \qquad y(0) = 40$$

$$40 = 300 + C \qquad C = -260$$

$$\boxed{y(t) = 300 - 260e^{-\frac{1}{150}t}}$$

$\lim_{t\to\infty} y = 300$ 300 lb salt in tank after a while





<u>example</u> Same setup. $y(0) = 40\,lb$, initially 600 gal of water

$\frac{1}{2}$ lb/gal, 4 gal/min in

flows out at 6 gal/min.

Find max amount of salt in tank.

$$\frac{dy}{dt} = 2 - \left(\frac{y}{\text{vol. of water}}\right)(6)$$

↳ $600 - 2t$

↳ 4 in, 6 out

net loss of 2 (negative)

$$y' = 2 - \frac{6y}{600-2t}$$

$$\boxed{y' = 2 - \frac{3y}{300-t}} \qquad y(0) = 40$$

$$y' + \frac{3}{300-t}y = 2$$

$$I = e^{\int \frac{3}{300-t}dt} = e^{-3\ln(300-t)} = e^{\ln(300-t)^{-3}}$$

$$I = \frac{1}{(300-t)^3}$$

$$\frac{1}{(300-t)^3}\left(y' + \frac{3}{300-t}y\right) = \frac{2}{(300-t)^3}$$

$$\frac{d}{dt}\left[\frac{1}{(300-t)^3}y\right] = \frac{2}{(300-t)^3}$$

$$\frac{1}{(300-t)^3}\, y = \frac{1}{(300-t)^2} + C$$

$$y = 300 - t + C\,(300-t)^3 \qquad y(0) = 40$$

$$40 = 300 + C\,(300)^3 \qquad C = -\frac{13}{135{,}000}$$

$$\boxed{y = 300 - t - \frac{13\,(300-t)^3}{135{,}000}}$$ (lb) amount of salt

$$\max y: \quad y' = 0 = -1 + \frac{39}{135000}(300-t)^2$$

$$t \approx 114 \quad \text{(min)}$$

$$\boxed{y_{max} = y(114) \approx 124}$$ (lb)

tank empty at $t = 300$

what is happening?

$$y' = \underbrace{2}_{\text{in}} - \underbrace{\frac{3y}{300-t}}_{\text{out}}$$

max y : rate in = rate out

initially, rate in > rate out

at $t = 114$, $y = 124$

rate in = rate out (max)

after $t = 114$, rate out > rate in