1.6 Substitution Method and Exact Equations

so far, we have seen : separable, first-order linear

but some are neither

<u>example</u> $\frac{dy}{dx} = x + y$ (linear, but let's pretend we didn't know)

let $v = x + y$ (y is function of x, so is v)

$$\frac{dv}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dv}{dx} - 1 = v$$

$$\frac{dv}{dx} = v + 1$$ separable in v and x

$$\frac{1}{v+1}\, dv = dx$$

$$\ln|v+1| = x + C$$

$v+1 = Ce^x$

$v = Ce^x - 1$ sub out v: $v = x+y$

$x+y = Ce^x - 1$

$$\boxed{y = Ce^x - 1 - x}$$

---

also works if $\frac{dy}{dx} = (x+y)^2$ $v = x+y$ also works

~~if~~ in fact, $\frac{dy}{dx} = f(ax+by+c)$

$v = ax+by+c$ always works

e.g. $\frac{dy}{dx} = (3x-2y+5)^4$

a first-order DE is said to be <u>homogeneous</u> if

$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ make substitution $v = \frac{y}{x}$

then transform into separable and/or first-order linear

<u>example</u> $y' = \frac{2x-y}{x+7y}$ not separable, not linear

divide top and bottom by x

$$y' = \frac{\frac{2x-y}{x}}{\frac{x+7y}{x}} = \frac{2-\left(\frac{y}{x}\right)}{1+7\left(\frac{y}{x}\right)} = f\left(\frac{y}{x}\right)$$ homogeneous

sub: $v = \frac{y}{x}$ eliminate y and y' from DE

$y = vx \rightarrow$ product of functions of

find y': $y' = v + xv'$

(product rule)

$$y' = \frac{2-\left(\frac{y}{x}\right)}{1+7\left(\frac{y}{x}\right)} \qquad v = \frac{y}{x} \qquad y' = v + xv'$$

$$v + xv' = \frac{2-v}{1+7v}$$

simplify

$$xv' = \frac{2-v}{1+7v} - v = \;= \frac{2-v}{1+7v} - \frac{v(1+7v)}{1+7v}$$

$$= \frac{2-v-v-7v^2}{1+7v}$$

$$x\frac{dv}{dx} = \frac{2-2v-7v^2}{1+7v}$$

separable in v and x

$$\frac{1+7v}{2-2v-7v^2}\,dv = \frac{1}{x}\,dx$$

$$\int \frac{1+7v}{2-2v-7v^2}\,dv = \int \frac{1}{x}\,dx$$

$$u = 2 - 2v - 7v^2$$

$$du = (-2 - 14v)\,dv = -2(1 + 7v)\,dv$$

$$-\frac{1}{2}\int \frac{1}{u}\,du = \int \frac{1}{x}\,dx$$

$$-\frac{1}{2}\ln|u| = \ln|x| + C$$

$$\ln|u| = -2\ln|x| + C$$

$$u = e^{-2\ln x} \cdot e^{C} = Cx^{-2}$$

$$2 - 2v - 7v^2 = Cx^{-2} \qquad v = \frac{y}{x}$$

$$2 - 2\cdot\frac{y}{x} - 7\cdot\frac{y^2}{x^2} = \frac{C}{x^2}$$

$$\boxed{2x^2 - 2xy - 7y^2 = C}$$

implicit solution

(explicit not possible)

Bernoulli differential equation

$$y' + P(x)\,y = Q(x)\,y^n \qquad n \neq 0,\ n \neq 1$$ (these make the DE separable or linear)

Substitution: $V = y^{1-n}$

then eliminate y' and y, solve for V, then undo subs.

example $x^2 y' + xy = y^2$

rewrite: $$y' + \underbrace{\frac{1}{x}}_{P(x)} y = \underbrace{\frac{1}{x^2}}_{Q(x)} y^{2}$$ $\leftarrow n$

$$V = y^{1-n} = y^{1-2} = y^{-1}$$

differentiate

$$V' = -y^{-2}\,y' \qquad \text{(chain rule)}$$

$$y' = -y^2 V'$$

$$y' + \frac{1}{x} y = \frac{1}{x^2} y^2$$

$$-y^2 v' + \frac{1}{x} y = \frac{1}{x^2} y^2$$

divide by $-y^2$

v from last page

$$v' - \frac{1}{x} y^{-1} = \frac{-1}{x^2}$$

$$v' - \frac{1}{x} v = -\frac{1}{x^2} \qquad \text{linear in } v$$

$$I = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = x^{-1}$$

$$x^{-1} v' - \frac{1}{x^2} v = -\frac{1}{x^3} \qquad \text{multiply by } I \text{ on both sides}$$

x^{-1}

$$\frac{d}{dx}\left(I v \right) = -\frac{1}{x^3}$$

integrate

$$x^{-1}v = \int -\frac{1}{x^3}\,dx$$

$$= \frac{1}{2x^2} + c$$

$$v = \frac{1}{2x} + cx$$

undo $v = y^{-1}$

$$y^{-1} = \frac{1+cx^2}{2x}$$

$$\boxed{y = \frac{2x}{1+cx^2}}$$