1.6 Subs methods and Exact Equations (continued)

from calculus $f(x,y) = c$ is a level curve
↳ a function of x

the total or exact differential of f is

$$df = \underline{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0} \quad \text{or} \quad \underline{f_x\,dx + f_y\,dy = 0}$$

and

$$\underline{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)} \quad \text{or} \quad \underline{f_{xy} = f_{yx}}$$

so, if we see a differential equation written in the form

$$\boxed{M\,dx + N\,dy = 0 \quad \text{and} \quad \cancel{M_y = N_y} \quad M_y = N_x}$$

Exact Differential Equation

if this is true, then there must be a function $f(x,y)$ such that $f_x = M$ and $f_y = N$

and the implicit solution is $f(x,y) = C$

<u>example</u> $\underbrace{(3x^2+2y^2)}_{M}dx + \underbrace{(4xy+6y^2)}_{N}dy = 0$

~~ex~~ exact if $M_y = N_x$

$M_y = 4y \qquad N_x = 4y$ so eq. is exact

So there is a function $f(x,y)$ such that

$$\left.\begin{aligned} f_x &= M = 3x^2+2y^2 \\ f_y &= N = 4xy+6y^2 \end{aligned}\right\}$$ pick one to integrate

$$\frac{\partial f}{\partial x} = 3x^2+2y^2 \rightarrow f = \int (3x^2+2y^2)\,d\underline{\underline{x}}$$

y is constant

$$f = x^3 + 2y^2x + g(y)$$

to find $g(y)$, take partial of f above with respect to y

and compare to N

$$f_y = \underbrace{4xy + g'(y)}_{\text{from } f \text{ we got by integration}} = \underbrace{4xy + 6y^2}_{N}$$

$g'(y) = 6y^2$ so $g(y) = 2y^3 + C_1$ C_1: constant

so $f(x,y) = x^3 + 2y^2x + 2y^3 + C_1$

Solution of this exact differential eq. is $f(x,y) = c$

$$f(x,y) = x^3 + 2y^2x + 2y^3 + C_1 = c$$

or $\boxed{x^3 + 2y^2x + 2y^3 = c}$

Some equations can be classified in many ways

$$\frac{dy}{dx} = -\frac{y}{x}$$

separable? yes $\frac{1}{y}\,dy = -\frac{1}{x}\,dx$

linear? yes $\frac{dy}{dx} + \frac{1}{x}y = 0$

homogeneous? yes $\frac{dy}{dx} = -\left(\frac{y}{x}\right) = f\left(\frac{y}{x}\right)$

exact? yes $x\,dy = -y\,dx$

$y\,dx + x\,dy = 0$

$M = y \qquad N = x$

$M_y = 1 \qquad N_x = 1$

if $\frac{dy}{dx} = \frac{y}{x}$ not exact anymore

$-y\,dx + x\,dy = 0$

$M = -y \qquad N = x$

$M_y = -1 \qquad N_x = 1 \qquad M_y \neq N_x$

Some 2nd-order eqs. can be transformed into 1st-order eqs.

1. the independent variable x does not appear

example: $y'' = -y$

make the substitution $\quad p = \dfrac{dy}{dx} = y'$

$$\frac{dp}{dx} = y'' = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$$

try it on $y'' = -y$

$$\underbrace{p\frac{dp}{dy}}_{y''} = -y$$

1st-order in p and y
and is separable
solve for y, then recover y
from $p = \dfrac{dy}{dx}$

$$p\,dp = -y\,dy$$

$$\frac{1}{2}p^2 = -\frac{1}{2}y^2 + C$$

$$p^2 = -y^2 + C$$

$$p = \sqrt{C - y^2} \qquad \text{(assume } p = y' > 0\text{)}$$

$$p = \frac{dy}{dx} = \sqrt{c - y^2} \qquad \text{separable in } y \text{ and } x$$

$$\underbrace{\frac{1}{\sqrt{c - y^2}}}_{\text{trig sub}} dy = dx$$

$$\vdots$$

$$y = \_\_\_$$

2. if the dependent variable y does not appear in eq.

example: $x y'' + y' = x$

Substitution: $p = \frac{dy}{dx} = y'$

$$\frac{dp}{dx} = y''$$

try it: $x \frac{dp}{dx} + p = x$ 1st-order in p and x

$$\frac{dp}{dx} + \frac{1}{x} p = 1$$

linear w/ p coefficients containing on x and/or numbers

$$I = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$x \left(\frac{dp}{dx} + \frac{1}{x} p \right) = x$$

$$x \frac{dp}{dx} + p = x$$

$$\frac{d}{dx}(xp) = x \quad \rightarrow \quad xp = \frac{1}{2}x^2 + C$$

$$p = \frac{1}{2}x + \frac{C}{x} = \frac{dy}{dx}$$

$$y = \int \left(\frac{1}{2}x + \frac{C}{x} \right) dx$$

constant

$$y = \frac{1}{4}x^2 + C \ln x + D$$