2.1 Population Models

Population $P(t)$, $P(t_0) = P_0$

the natural / exponential model: $\frac{dP}{dt} = kP$ rate proportional to size

high $P \to$ fast growth

low $P \to$ slow growth

$\frac{dP}{dt} = kP$, $P(t_0) = P_0$ is separable

solution is $P(t) = P_0 e^{kt}$ no limit to how high P can be

short term is ok but not realistic for long term.

let's extend the model: $\frac{dP}{dt} = (\underbrace{\beta(t)}_{\text{birth}} - \underbrace{\delta(t)}_{\text{death}})P$

notice if β, δ are constants and

$\beta > \delta \to$ natural model

(natural: k as constant net growth rate) let's model the birth rate as a decreasing linear function of P

$$\beta(t) = \beta_0 - \beta_1 P \qquad \beta_0, \beta_1 \text{ constants}$$

(can model the decreasing food as population grows)

death rate is the same as before: $\delta(t) = \delta_0$ constant

sub into $\dfrac{dP}{dt} = (\beta(t) - \delta(t))P$

$$= (\beta_0 - \beta_1 P - \delta_0)P$$

$$= (\beta_0 - \delta_0)P - \beta_1 P^2$$

$$\frac{dP}{dt} = aP - bP^2$$

if $a, b > 0$ then this equation is called the <u>logistic equation</u>

$$\frac{dP}{dt} = bP\left(\frac{a}{b} - P\right)$$

$$\frac{dP}{dt} = kP(M - P)$$

$M = \frac{a}{b} \rightarrow$ carrying capacity

this is the cap to the population

<u>example</u> A population of squirrels satisfies the logistic equation

$$\frac{dP}{dt} = aP - bP^2$$

If the initial population is 120, and there are 8 births per month and 6 deaths per month at $t = 0$.

Find the limiting population and the time to reach 95% of that number.

$$\frac{dP}{dt} = \underbrace{aP}_{\text{net birth rate}} - \underbrace{bP^2}_{\text{death rate}} \qquad a > 0,\ b > 0$$

at $t=0$, $P(0)=120$

8 births/mo $\rightarrow$ $\frac{dP}{dt} = 8 = aP(0) = a(120)$

$$a = \frac{8}{120} = \frac{1}{15}$$

6 deaths/mo $\rightarrow$ $\frac{dP}{dt} = 6 = b[P(0)]^2 = b(120)^2$

$$b = \frac{6}{(120)^2} = \frac{1}{2400}$$

$$\frac{dP}{dt} = \frac{1}{15}P - \frac{1}{2400}P^2$$

$$= \frac{1}{2400}P\left(\frac{1/15}{1/2400} - P\right) = \frac{1}{2400}P(160-P)$$

$$KP(M-P)$$

so, the carrying capacity (limiting population) is 160.

let's solve $\frac{dP}{dt} = \frac{1}{2400} P(160-P)$ $\quad P(0)=120$

let's solve it as a separable

$$\int \underbrace{\frac{1}{P(160-P)}}_{\text{partial fraction expansion}} dp = \int \frac{1}{2400} dt$$

$$\frac{1}{P(160-P)} = \frac{A}{P} + \frac{B}{160-P}$$

$$1 = A(160-P) + BP$$

$$0P + 1 = (B-A)P + 160A$$

$$B-A=0$$

$$160A = 1$$

$$A = \frac{1}{160} \quad \text{so} \quad B = \frac{1}{160}$$

$$\int \left(\frac{1}{160}\frac{1}{P} + \frac{1}{160}\frac{1}{160-P} \right) dp = \int \frac{1}{2400} dt$$

$$\frac{1}{160}\ln P - \frac{1}{160}\ln(160-P) = \frac{1}{2400}t + C$$

$$\ln P - \ln(160-P) = \frac{1}{15}t + C$$

$$\ln\left(\frac{P}{160-P}\right) = \frac{1}{15}t + C$$

$$\frac{P}{160-P} = e^{\frac{1}{15}t+c} = e^{\frac{1}{15}t}\cdot e^{c} = Ce^{\frac{1}{15}t}$$

$P(0)=120$

$$P = (160-P)Ce^{\frac{1}{15}t}$$

$$P = 160Ce^{\frac{1}{15}t} - PCe^{\frac{1}{15}t}$$

$$P + PCe^{\frac{1}{15}t} = 160Ce^{\frac{1}{15}t}$$

$$P\left(1+Ce^{\frac{1}{15}t}\right) = 160Ce^{\frac{1}{15}t}$$

$$P = \frac{160Ce^{\frac{1}{15}t}}{1+Ce^{\frac{1}{15}t}}$$

$$120 = \frac{160C}{1+C} \rightarrow C = 3$$

$$P(t) = \frac{480e^{\frac{1}{15}t}}{1+3e^{\frac{1}{15}t}}$$

$$P(t) = \frac{480}{3+e^{-\frac{1}{15}t}}$$

find t such that

$$P = (0.95)(160)$$

$$(0.95)(160) = \frac{480}{3+e^{-\frac{1}{15}t}}$$

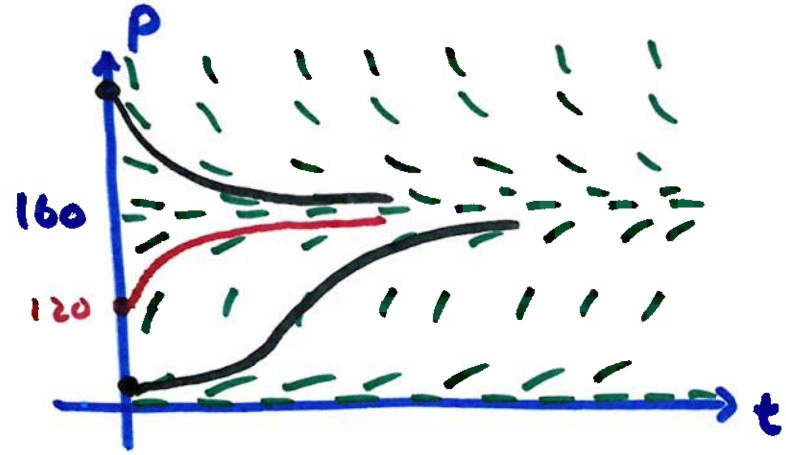
$\vdots$

$$e^{-\frac{1}{15}t} = \frac{480}{(0.95)(160)} - 3$$

$\vdots$

$t \approx 28$ months

$$\frac{dP}{dt} = \frac{1}{2400}P(160-P)$$



$P' = 0 \rightarrow P = 0,\ P = 160$

$P > 160 \rightarrow P' < 0$, more steep as P increases

$0 < P < 160 \rightarrow P' > 0$, flat near $P = 0, 160$
steep in between