2.2 Equilibrium Solutions and Stability

autonomous differential eq: $\dfrac{dy}{dt} = \underbrace{f(y)}_{\text{no } t \text{ (or } x\text{)}}$

(always separable)

population models are autonomous: $\dfrac{dy}{dt} = ky$

$$\frac{dy}{dt} = ky(M-y)$$

the values of y where $f(y)=0$ (right side is zero) are called critical points

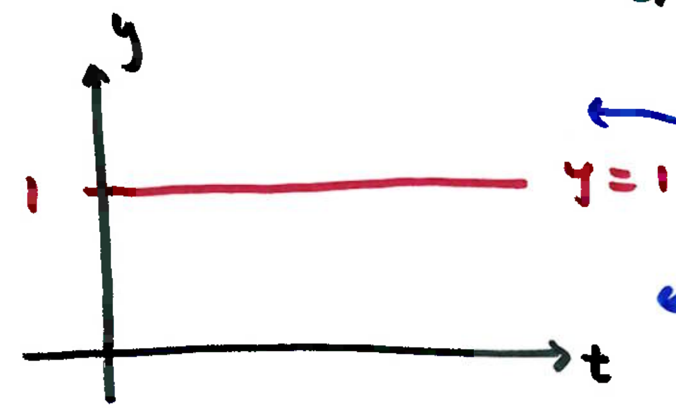
$\dfrac{dy}{dt} = y-1$ has critical pt $y=1$

if C is a critical pt, then $y=C$ is called an

equilibrium solution

$\frac{dy}{dt} = y-1$ has an equilibrium solution $y=1$

↳ $y'=0$ so y stays the same



above or below $y=1$ solutions change

Some equations are easy to solve

$$\frac{dy}{dt} = y-1 \qquad y(0) = y_0$$

$$\frac{1}{y-1}\,dy = dt$$

$$\ln(y-1) = t + c$$

$$y - 1 = Ce^{t}$$

$$y = 1 + Ce^{t} \qquad y(0) = y_0$$

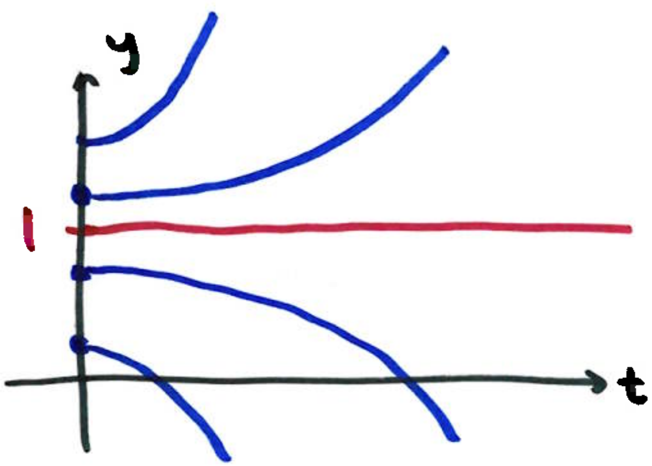
$$y_0 = 1 + C \quad \text{so} \quad C = y_0 - 1$$

$$\boxed{y = 1 + (y_0 - 1)e^{t}}$$

if $y_0 = 1$, then $y = 1$ for all t

if $y_0 > 1$, then as $t \to \infty$, $y \to \infty$

if $y_0 < 1$, then as $t \to \infty$, $y \to -\infty$



→ solutions nearby go <u>away</u> from this equilibrium

this is called an <u>unstable</u> equilibrium if equation is not easy to solve, we can still understand the stability of equilibrium solutions by using a <u>phase diagram</u> w/o solving the differential eq.

$$\frac{dy}{dt} = y - 1 \qquad \text{equilibrium} : \; y = 1 \quad \left(\frac{dy}{dt} = 0\right)$$

then identify $\frac{dy}{dt}$ near each critical pt

solutions here increase over time

$\frac{dy}{dt} = y - 1$

if $y < 1$

$y' < 0$

(decreasing)

$y' = 0$

(equilibrium)

solutions go away from $y = 1$

so it is <u>unstable</u>

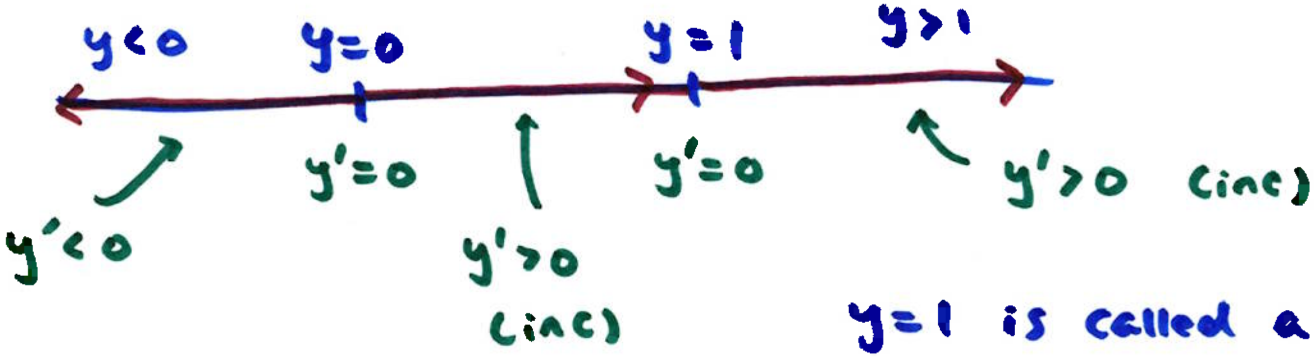
$\frac{dy}{dt} = y - 1$ and if $y > 1$

then $y' > 0$

(increasing)

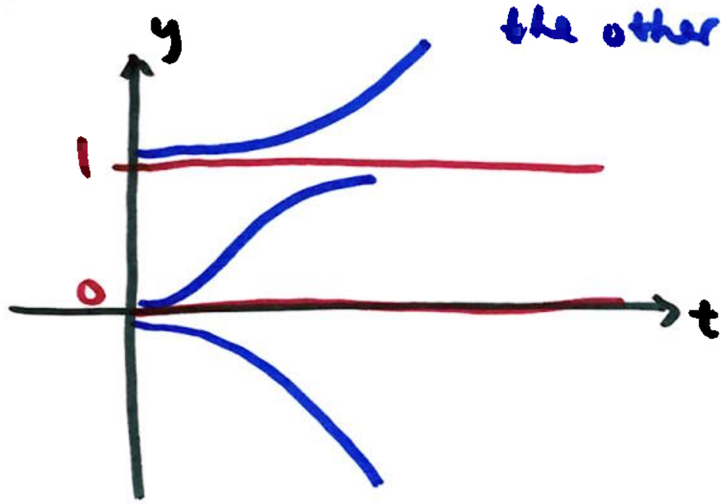
$$\frac{dy}{dt} = y(y-1)^2$$

Critical pts: $y=0$, $y=1$



$y=1$ is called a <u>semi-stable</u> equilibrium (one side approaches the other leaves)

$y=0$ is <u>unstable</u>



$$\frac{dy}{dt} = (y-1)(y+2)(y^2-9)$$

Critical pts: $y = 1, -2, -3, 3$