2.2 Equilibrium solutions and stability

autonomous differential eqs: the right side does NOT depend on the independent variable (t or x)

for example, $\frac{dy}{dt} = ky$ no t explicitly on the right

$$\frac{dy}{dt} = ky(M-y)$$

the values of y where $\frac{dy}{dt} = 0$ are called critical points

and the solution $y = C$ where C is a critical point is an equilibrium solution

for example, $\frac{dy}{dt} = ky(M-y)$ has critical pts at $y=0$,
$y = M$

if $\underbrace{y=0 \text{ or } y=M}_{\text{equilibrium solutions of } \frac{dy}{dt}=ky(M-y)}$, $\frac{dy}{dt}=0$ so y remains constant

initial condition plays an important part in the long-term behavior.

for example, $\frac{dy}{dt} = y-1$

separable
linear
autonomous

critical pt: $y-1=0 \rightarrow y=1$

equilibrium solution $y=1$

let $y(0)=y_0$

$$\frac{1}{y-1}\,dy = dt$$

$$\ln(y-1) = t + C$$

$$y - 1 = Ce^{\text{[redacted]}t}$$

$$y = 1 + Ce^{\text{[redacted]}t} \qquad y(0) = y_0$$

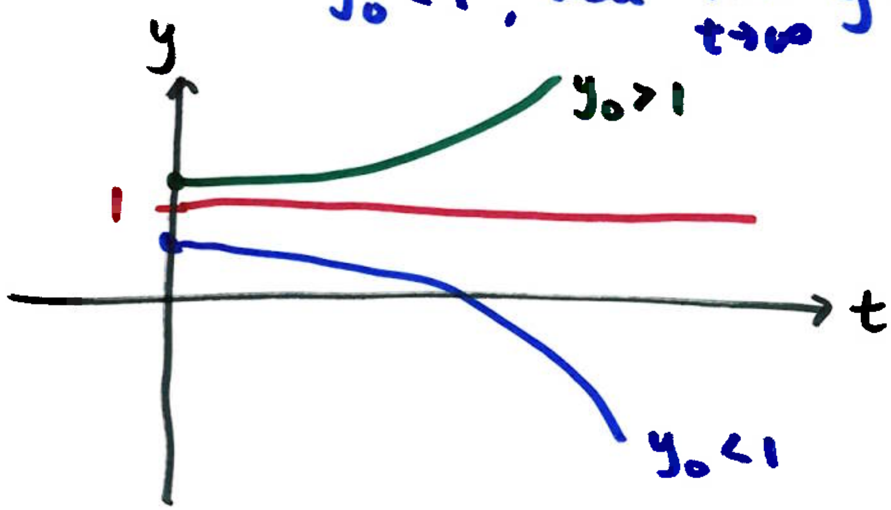
$$y_0 = 1 + C \qquad C = y_0 - 1$$

$$\boxed{y(t) = 1 + (y_0 - 1)e^{\text{[redacted]}t}}$$

notice $y_0 = 1$, then $y = 1$ for all t (equilibrium)

$y_0 > 1$, then $\lim_{t \to \infty} y = \infty$

$y_0 < 1$, then $\lim_{t \to \infty} y = -\infty$



any small change from $y_0 = 1$ drives the solution to ∞ or $-\infty$ (away from equilibrium)

→ this is an unstable equilibrium

we may not want to solve the differential eq. to determine stability

→ we can use a phase diagram to determine stability

$$\frac{dy}{dt} = y - 1$$

critical point : $y = 1$

analyze y' to the right and left

$y' < 0$ (decreasing)

$y' = 0$

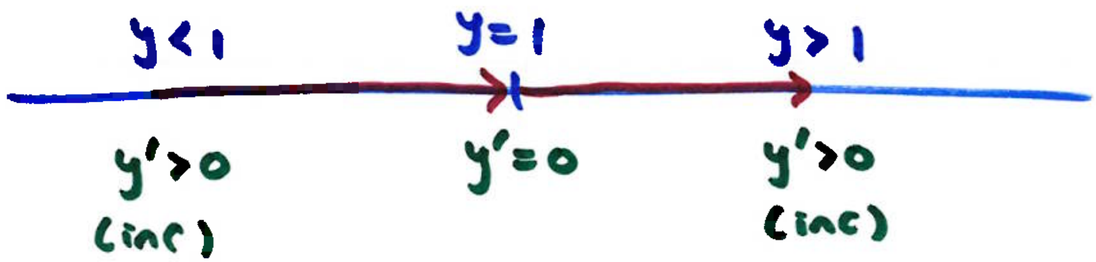
$y' => 0$ (increasing)

($y' = y - 1$ so $y' > 0$ if $y > 1$)

solutions go away from equilibrium
unstable

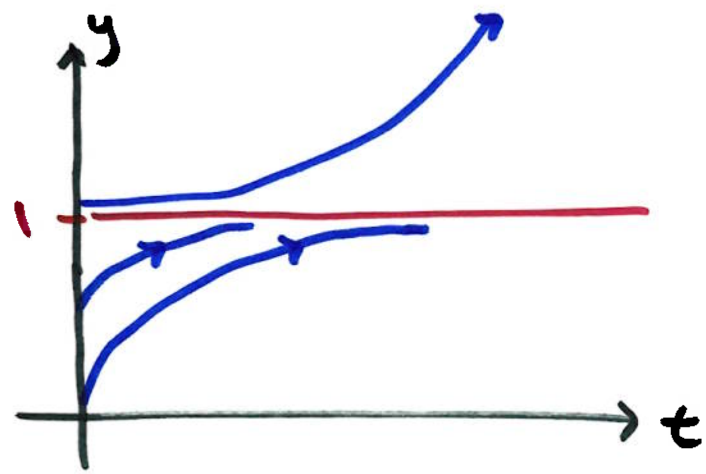
another example, $\frac{dy}{dt} = (y-1)^2$

critical pt: $y = 1$



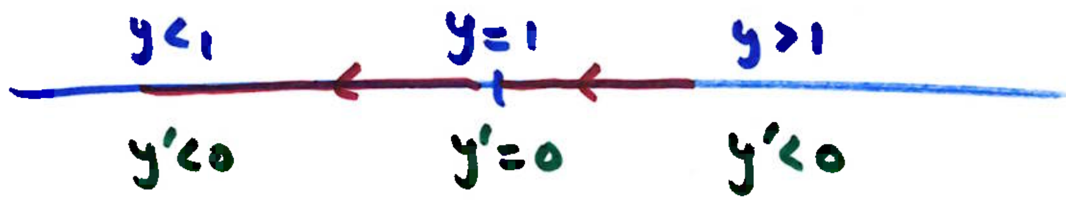
this is a **semi-stable** equilibrium

using this, we know the solutions will look like



$$\frac{dy}{dt} = -(y-1)^2$$

critical pt $y=1$, situation near critical pt is the opposite

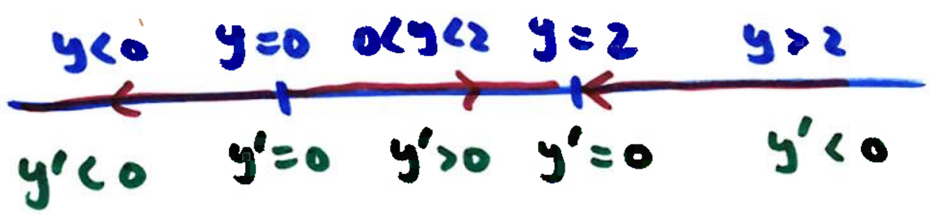


Still semi-stable

another example

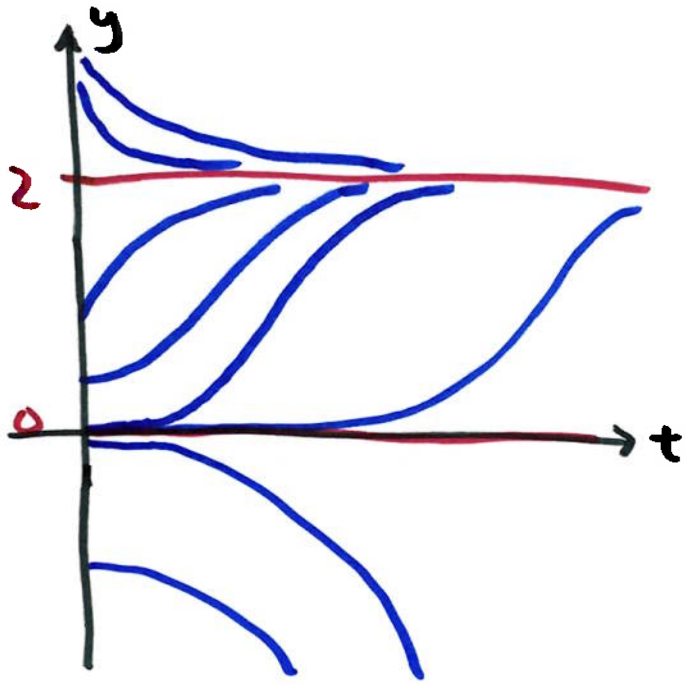
$$\frac{dy}{dt} = y(2-y) = ky(M-y)$$

Critical pts: $y=0$, $y=2$



$y=2$ is asymptotically stable or just stable

$y=0$ is unstable



example

$$\frac{dy}{dt} = (y-1)(y-2)(y^2-9)$$

critical pts : 1, 2, 3, -3

$y=-3$ stable

$y=1$ unstable

$y=2$ stable

$y=3$ unstable