2.4 Numerical Approx. – Euler's Method

$y' = f(x,y)$ ~~$y(\hat{x}) = y_0$~~ $y(x_0) = y_0$

"solve" → find y = function of x → bunch of points

(x_0, y_0)
(x_1, y_1)
(x_2, y_2)
$\vdots$

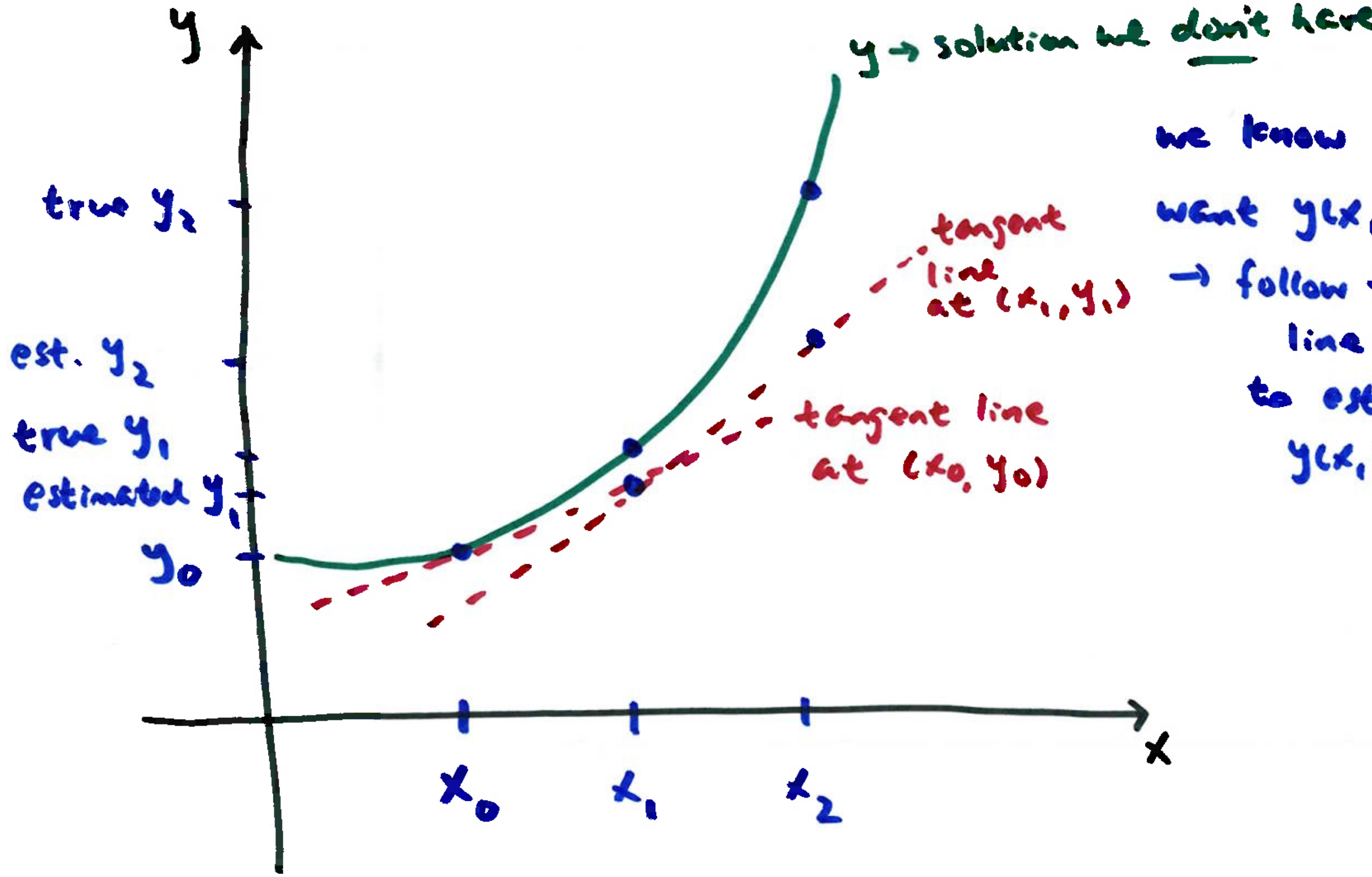
if equation is separable, linear, homogeneous, or exact we can solve exactly

but if not or if we don't want to solve using the techniques (e.g. integration too messy), we can use numerical methods to find y given an x.

Euler's method (tangent line method)

$y' = f(x,y)$ want y, we know its slope at (x,y)

$y(x_0) = y_0$



we know $y(x_0) = y_0$

want $y(x_1)$

$\rightarrow$ follow tangent line at (x_0, y_0) to estimate $y(x_1) \approx y_1$

Euler's method algorithm

given: $y' = f(x,y)$ find: $y(x)$

(x_0, y_0) initial condition specified

decide step size: interval between x's → call it h

build tangent line at (x_0, y_0)

$$y - y_0 = f(x_0,y_0)(x - x_0) \longrightarrow y = y_0 + f(x_0,y_0)h$$

this gives y_1: $\boxed{y_1 = \underbrace{y_0}_{\text{previous } y} + \underbrace{f(x_0,y_0)}_{\text{slope at previous point}}\,\overbrace{h}^{\text{step size}}}$

if x_1 is not the target x, repeat

generalize: $\cancel{x_n = x_{n-1}}$

$$\boxed{\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + f(x_n, y_n)h \end{aligned}}$$

iterate until x_{n+1} = target x

<u>example</u> $y' = \overbrace{2y-3x}^{f(x,y)} \qquad y(0)=1$

estimate $y(0.5)$ using step size $h = 0.25$

$x_0 = 0$

$y_0 = 1$

$x_1 = x_0 + h = 0 + 0.25 = 0.25$

$y_1 = y_0 + f(x_0, y_0)h$

$= 1 + [\underbrace{2(1) - 3(0)}_{\text{use "old" } x,y}](0.25) = 1.5$

x_1 is not yet the target x (0.5), so repeat

$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$ (target, so stop after this iteration)

$y_2 = y_1 + f(x_1, y_1)h$

$= 1.5 + [2(1.5) - 3(0.25)](0.25) = \boxed{2.0625}$

estimate of $y(0.5)$

how good is the estimate?

let's solve $y' = 2y - 3x$, $y(0) = 1$ exactly and compare

linear
homogeneous

$y' - 2y = -3x \qquad I = e^{\int -2\,dx} = e^{-2x}$

$e^{-2x}(y' - 2y) = -3x e^{-2x}$

$\frac{d}{dx}(e^{-2x} y) = -3x e^{-2x}$

$e^{-2x} y = \int -3x e^{-2x}\,dx$ (by parts)

$\vdots$

$y = \frac{3}{4}(2x+1) + \frac{1}{4}e^{2x}$

true $y(0.5) = 2.1796$ (estimate was 2.0625)

to improve estimate, use more steps or smaller step size

if we had used $h = 0.01$ (50 steps) $y(0.5) \approx 2.1729$

generally, smaller $h \rightarrow$ better estimate (more steps)

usually we don't know the true y (if we had it why estimate?)

$\rightarrow$ how do we know if the estimate is good?

if further refinement of step size does not result in significant changes

$y' = 2y - 3x$, $y(0) = 1$, estimate $y(1)$

$h = 0.5$, $y(1) = 3.25$

$h = 0.05$, $y(1) = 3.932$

significant change, suggesting estimate can be improved

$h = 0.01$, $y(1) = 4.061$

still changing, improve further

$h = 0.001$, $y(1) = 4.094$

good enough?

$h = 0.0005$, $y(1) = 4.095$

settling down, suggesting the estimate is good