3.2 Matrices and Gaussian Elimination

matrix: array of numbers

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

rows: 2
columns: 2
2 by 2 matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

2 × 3 matrix

from last time: $x_1 + 3x_2 = 9$ — ①
$2x_1 + x_2 = 8$ — ②

solve by elimination: $-2①+②$

$-5x_2 = -10 \quad x_2 = 2$

from ① $\quad x_1 = 9 - 3x_2$
$= 3$

represent the same system by this coefficient matrix

$$\begin{array}{cccc} x_1 & x_2 & \text{right side} & \\ \left[\begin{matrix} 1 \\ 2 \end{matrix}\right. & \begin{matrix} 3 \\ 1 \end{matrix} & \left.\begin{matrix} 9 \\ 8 \end{matrix}\right] & \begin{matrix} \text{eq. 1} \\ \text{eq. 2} \end{matrix} \end{array}$$

elimination in matrix representation is made up of

elementary row operations

- swap any two rows
- multiply one row by a non-zero constant
- multiply one row by a non-zero constant and add to another row

① $x_1 + 3x_2 = 9$

② $2x_1 + x_2 = 8$

$$\longrightarrow \begin{bmatrix} 1 & 3 & 9 \\ 2 & 1 & 8 \end{bmatrix}$$

multiply ① by -2

add to ②

$$\xrightarrow{(-2)R_1 + R_2} \begin{bmatrix} 1 & 3 & 9 \\ 0 & -5 & -10 \end{bmatrix}$$

$$\xrightarrow{(-\frac{1}{5})R_2} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \text{right side} \end{array} \\ \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 2 \end{bmatrix} \end{array}$$

Solution: read bottom up

row 2: $0 \cdot x_1 + 1 \cdot x_2 = 2$ so, $\boxed{x_2 = 2}$

row 1: $1 \cdot x_1 + 3 \cdot x_2 = 9$

$$x_1 = 9 - 3x_2 = 9 - 3(2) = 3 \quad \boxed{x_1 = 3}$$

left most non-zero

back to $\begin{bmatrix} \boxed{1} & 3 & 9 \\ 0 & \boxed{-5} & -10 \end{bmatrix}$ this matrix is in

row echelon form

"stairs"

the left most non-zero number (called "pivot") of a row is below and to the right of the pivot of the row above pivot row 1

$$\begin{bmatrix} \boxed{1} & 2 \\ \boxed{3} & 4 \end{bmatrix}$$ is NOT in row echelon form

pivot row 2 is NOT to the right and below the pivot of the row above it

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{4} & 5 \end{bmatrix}$$ is in row echelon form

pivot of row 1

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{5} \\ 0 & 0 & 0 \end{bmatrix}$$

pivot of row 2 it is below and to the right of row pivot of row 1

row 3 does NOT have a pivot

so, this is in row echelon form

→ in row echelon form, a row of all zeros is at the bottom

also, notice all numbers below a pivot are zeros

Solving a system → put coefficient matrix into row echelon form

the process of making a matrix in row echelon form is called row reduction or Gaussian elimination

example

$$\begin{aligned} x_2 + 4x_3 &= -3 \\ x_1 + 3x_2 + 6x_3 &= 4 \\ 2x_1 + 5x_2 + 8x_3 &= 5 \end{aligned}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \text{right} \end{array}$$
$$\begin{bmatrix} 0 & 1 & 4 & -3 \\ 1 & 3 & 6 & 4 \\ 2 & 5 & 8 & 5 \end{bmatrix}$$

sometimes called the augmented matrix because right side numbers are included

put into row echelon form

swap rows 1 and 2 (or 1 and 3) so we have a pivot on the upper left

$$\xrightarrow{\text{swap}(R_1, R_2)} \begin{array}{c} \text{pivot} \\ \begin{bmatrix} \boxed{1} & 3 & 6 & 4 \\ 0 & 1 & 4 & -3 \\ 2 & 5 & 8 & 5 \end{bmatrix} \end{array}$$

every # below pivot should be zeros

make it 0

$$\xrightarrow{(-2)R_1 + R_3} \begin{bmatrix} \boxed{1} & 3 & 6 & 4 \\ 0 & \boxed{1} & 4 & -3 \\ 0 & -1 & -4 & -3 \end{bmatrix}$$

pivot

pivot

zeros

make 0

$$\xrightarrow{R_2 + R_3} \begin{array}{cccc} x_1 & x_2 & x_3 & \text{right} \end{array}$$

$$\begin{bmatrix} \boxed{1} & 3 & 6 & 4 \\ 0 & \boxed{1} & 4 & -3 \\ 0 & 0 & 0 & \boxed{-6} \end{bmatrix}$$

row echelon form

Solve: row 3 $\rightarrow 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = -6$

$0 = -6$ false!

So, there is <u>no</u> solution

<u>example</u>

$$x_1 - x_2 + x_3 = 7$$
$$3x_1 + 2x_2 - 12x_3 = 11$$
$$4x_1 + x_2 - 11x_3 = 18$$

$$\begin{bmatrix} \boxed{1} & -1 & 1 & 7 \\ 3 & 2 & -12 & 11 \\ 4 & 1 & -11 & 18 \end{bmatrix}$$

$$\xrightarrow[(-4)R_1+R_3]{(-3)R_1+R_2} \begin{bmatrix} \boxed{1} & -1 & 1 & 7 \\ 0 & \boxed{5} & -15 & -10 \\ 0 & 5 & -15 & -10 \end{bmatrix}$$

make 0

$$\xrightarrow{(-1)R_2+R_3} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \begin{bmatrix} \boxed{1} & -1 & 1 & 7 \\ 0 & \boxed{5} & -15 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

row echelon form

Solve: row 3 $\rightarrow$ $0 = 0$ $\rightarrow$ infinite solutions

one of the variables is arbitrary (free variable)

the variable w/o pivot in its column is chosen to be free

here, x_3 is free

$$x_3 = t$$

row 2 $\rightarrow$ $5x_2 - 15x_3 = -10$

$$x_2 = 3x_3 - 2 = 3t - 2$$

row 1 $\rightarrow$ $\cdots \rightarrow$ $x_1 = 5 + 2t$