3.2 Reduced Row Echelon Form

row echelon: reduce by row operations (Gaussian elimination)

all numbers below a pivot (leftmost non zero in a row) are zero

$$\begin{bmatrix} 1 & 3 & 9 \\ 2 & 1 & 8 \end{bmatrix} \xrightarrow{(-2)R_1+R_2} \begin{bmatrix} \boxed{1} & 3 & 9 \\ 0 & \boxed{-5} & -10 \end{bmatrix}$$

pivots

#'s below are zero

reduced row echelon: row echelon AND every pivot is the only non-zero # in its column and every pivot 1

$$\begin{bmatrix} 1 & 3 & 9 \\ 0 & -5 & -10 \end{bmatrix}$$

$$\xrightarrow{(-\frac{1}{5})R_2} \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{(-3)R_2+R_1} \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & 2 \end{bmatrix}$$

reduced row echelon form (RREF)

$x_1 + 3x_2 = 9$

$2x_1 + x_2 = 8$

row echelon for a matrix is NOT unique

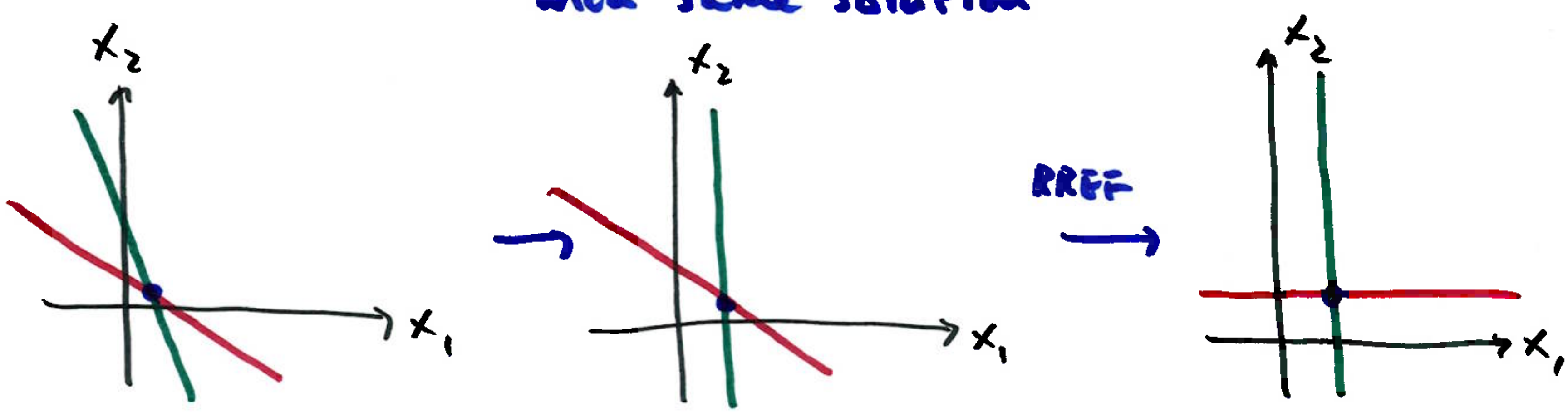
$$\begin{bmatrix} 1 & 3 & 9 \\ 2 & 1 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 9 \\ 0 & -5 & -10 \end{bmatrix}$$ one row echelon

another: $$\begin{bmatrix} 2 & 1 & 8 \\ 1 & 3 & 9 \end{bmatrix} \xrightarrow{(-\frac{1}{2})R_1 + R_2} \begin{bmatrix} 2 & 1 & 8 \\ 0 & 5/2 & 5 \end{bmatrix}$$

Another row echelon

solution is the same!

row reduction $\rightarrow$ replacing a system w/ another system with same solution



row echelon : change lines but keep intersection
(infinitely many possibilities)

rref : vertical and horizontal lines only (only one possibility)

rref for a matrix <u>is</u> unique

$$\begin{bmatrix} 1 & 3 & 9 \\ 2 & 1 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 9 \\ 0 & -5 & -10 \end{bmatrix} \to \boxed{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & 1 & 8 \\ 0 & 5/2 & 5 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 8 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\to \boxed{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}}$$

rref → no back substitution necessary (read rows for solution)

<u>example</u>

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 12 \\ 2x_1 + 5x_2 + 2x_3 &= 12 \\ 2x_1 + 7x_2 + 7x_3 &= 43 \end{aligned}$$

$$\begin{bmatrix} \boxed{1} & 3 & 2 & 12 \\ 2 & 5 & 2 & 12 \\ 2 & 7 & 7 & 43 \end{bmatrix}$$

make it rref

Gauss-Jordan elimination

$$\xrightarrow[(-2)R_1+R_3]{(-2)R_1+R_2} \begin{bmatrix} 1 & 3 & 2 & 12 \\ 0 & -1 & -2 & -12 \\ 0 & 1 & 3 & 19 \end{bmatrix}$$

$$\xrightarrow{R_2+R_3} \begin{bmatrix} 1 & 3 & 2 & 12 \\ 0 & -1 & -2 & -12 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\xrightarrow{(3)R_2+R_1} \begin{bmatrix} 1 & 0 & -4 & -24 \\ 0 & -1 & -2 & -12 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\xrightarrow[(4)R_3+R_1]{(2)R_3+R_2} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 7 \end{bmatrix} \xrightarrow{(-1)R_2} \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

row 3 : $x_3 = 7$

row 2 : $x_2 = -2$

row 1 : $x_1 = 4$

example

$$\begin{aligned} x_1 + 3x_2 - 15x_3 + 7x_4 &= 0 \\ x_1 + 4x_2 - 19x_3 + 10x_4 &= 0 \\ 2x_1 + 5x_2 - 26x_3 + 11x_4 &= 0 \end{aligned}$$

right side all zeros

homogeneous system

has nothing to do with $v = \frac{y}{x}$

$$\begin{bmatrix} 1 & 3 & -15 & 7 & 0 \\ 1 & 4 & -19 & 10 & 0 \\ 2 & 5 & -26 & 11 & 0 \end{bmatrix}$$

$$\xrightarrow[(-2)R_1 + R_3]{(-1)R_1 + R_2} \begin{bmatrix} 1 & 3 & -15 & 7 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & -1 & 4 & -3 & 0 \end{bmatrix}$$

$$\xrightarrow[R_3 + R_3]{(-3)R_2 + R_1} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & -3 & -2 & 0 \\ 0 & \boxed{1} & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

RREF

row 3 : $0 = 0$ infinitely-many solutions

→ free variable(s)

variables w/ no pivots in their columns are free

here, x_3 and x_4 are free

$x_4 = r$

$x_3 = t$

$x_2 - 4x_3 + 3x_4 = 0 \rightarrow x_2 = 4t - 3r$

$x_1 - 3x_3 - 2x_4 = 0 \rightarrow x_1 = 3t + 2r$