3.5 Matrix Inverse

scalar: $a \cdot a^{-1} = a^{-1} \cdot a = 1$ for $a \neq 0$ $a^{-1} = \frac{1}{a}$

matrix: $AA^{-1} = A^{-1}A = I$ ← identity matrix

inverse of A

I: square matrix w/ 1 on its main diagonal and 0 everywhere else

2x2 identity: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4x4 identity: $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

any square matrix A

$AI = IA = A$

for example, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$AI = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

notice $AA^{-1} = A^{-1}A = I$ → only square A can have A^{-1}

but <u>not all</u> square A have A^{-1}

(just like not all scalar a have a^{-1})

if A^{-1} exists, then we say A is <u>invertible</u>

if not, A is not invertible or A is <u>singular</u>

for 2×2 A, A^{-1} has a very simple formula:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$ad-bc$ is the <u>determinant</u> of A

($ad-bc = 0$ → A is not invertible)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^{-1} = \frac{1}{1\cdot 4 - 3\cdot 2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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inverse can be used to solve systems

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_1 + 4x_2 &= 6 \end{aligned} \longrightarrow \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}}_{A\vec{x} = \vec{b}}$$

Solution: Gaussian elimination $\rightarrow$ always possible

or $A\vec{x} = \vec{b}$ so $\vec{x} = \underbrace{A^{-1}\vec{b}}_{\text{order is important}}$ $\rightarrow$ only if A^{-1} exists here, $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$ (previous example)

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ 9/2 \end{bmatrix} \quad \begin{matrix} x_1 = -4 \\ x_2 = 9/2 \end{matrix}$$

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for $n \times n$ A where $n > 2$, no nice formulas exist for A^{-1} but there is a general algorithm that works for all n (including $n = 2$)

given $n \times n$ A, make an augmented matrix with left half being A and right half I of the same size

$$\Big[\underbrace{A}_{n \times n} \;\vdots\; \underbrace{I}_{n \times n}\Big]$$ this matrix is $n \times 2n$

$\longrightarrow$ row operations $\longrightarrow$ $\left[I \;\vdots\; A^{-1} \right]$

try it on $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(-3)R_1 + R_2} \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{(-\frac{1}{2})R_2} \left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

A^{-1}

try $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[(-1)R_1 + R_3]{(-1)R_1 + R_2} \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(-1)R_2 + R_3} \left[\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I} \; \vdots \; \underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{array}}_{A^{-1}}\right]$$

if A^{-1} does not exist, then we won't be able to make the left half into an I

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$$

$$\left[\begin{array}{cccccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & x & x & x \end{array}\right]$$

A is not invertible

no pivot in row 3

not possible to become I

why does $[A \;\; I] \to \cdots \to [I \;\; A^{-1}]$ work?

revisit

$$a_1 x_1 + b_1 x_2 = c_1$$
$$a_2 x_1 + b_2 x_2 = c_2$$

to find x_1, x_2, we do $\left[\begin{array}{cc:c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}\right]$

$$\to \cdots \to \left[\begin{array}{cc:c} 1 & 0 & x_1 \\ 0 & 1 & x_2 \end{array}\right]$$

for inverse of A, the same principle is used

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\to \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} f \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

two columns together

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

solving first system twice, & one for each column but together (columns do not change using row ops)

$$\to \cdots \to \left[\, I \;\vdots\; A^{-1} \,\right]$$