3.6 Determinants

matrix A $(n \times n)$, its determinant is $\det A$ or $\det(A)$ or $|A|$

last time, we saw that for 2×2 A, $A^{-1} = \frac{1}{\underbrace{ad-bc}_{\text{determinant}}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

for 2×2 A, $\det A = \{ ad-bc$ if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(not as simple for $n > 2$)

$\det A$ tells us a lot of things, one of them is the invertibility of A

→ if $\det A = 0$, then A^{-1} does not exist

↳ to calculate $\det A$ ($n\times n$ especially if $n > 2$), we can use the cofactor expansion

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

the <u>minor</u> of a matrix M_{ij} (i^{th} row, j^{th} col) is the matrix that remains after crossing out the i^{th} row and j^{th} column

$$M_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} \qquad M_{31} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$$

1st row   2nd column

cofactor of i^{th} row, j^{th} column is $C_{ij} = (-1)^{i+j} |M_{ij}|$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = (-1)(36-42) = 6$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = (1)(12-15) = -3$$

$(-1)^{i+j}$ alternates signs of cofactor this way

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

for finding $\det A$, we sum up $a_{ij} C_{ij}$ along any row or column

cofactor signs

example

$$A = \begin{bmatrix} 2^{+} & 0^{-} & 4^{+} \\ 3^{-} & 4^{+} & 2^{-} \\ 0^{+} & 4^{-} & -2^{+} \end{bmatrix}$$

pick ANY row or column to expand

as an example, we choose column 1

$$\det = \underbrace{(2)}_{a_{11}} \underbrace{\begin{vmatrix} 4 & 2 \\ 4 & -2 \end{vmatrix}}_{C_{11}} - (3) \begin{vmatrix} 0 & 4 \\ 4 & -2 \end{vmatrix} + (0) \begin{vmatrix} 0 & 4 \\ 4 & 2 \end{vmatrix}$$

$$= (2)(-8-8) - (3)(0-16) + (0)(0-16)$$

$$= 16$$

$$A = \begin{bmatrix} 2^{+} & 0^{-} & 4^{+} \\ 3^{-} & 4^{+} & 2^{-} \\ 0^{+} & 4^{-} & -2^{+} \end{bmatrix}$$

pick <u>row 2</u>

$$\det A = -(3)\begin{vmatrix} 0 & 4 \\ 4 & -2 \end{vmatrix} + (4)\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} - (2)\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix}$$

$$= (-3)(-16) + (4)(-4) - (2)(8)$$

$$= 16$$

det of 3×3 $\rightarrow$ sum of a bunch of det of 2×2

det of 4×4 $\rightarrow$ " " det of 3×3

which are bunch of det of 2×2

pick row/col w/ many zeros

example

$$A = \begin{bmatrix} 8^{+} & 0^{-} & 0^{+} & 5^{-} \\ 5^{-} & 8^{+} & 3^{-} & -7^{+} \\ 2^{+} & 0^{-} & 0^{+} & 0^{-} \\ 7^{-} & 2^{+} & 1^{-} & 7^{+} \end{bmatrix}$$

row 3 is good b/c it has most zeros
(even though ANY row/col is ok)

$$\det A = \underbrace{(2)\begin{vmatrix} 0^{+} & 0^{-} & 5^{+} \\ 8 & 3 & -7 \\ 2 & 1 & 7 \end{vmatrix}}_{\text{cofactor exp like last example (row 1 is good)}} + (0)\begin{vmatrix} \text{I.D.C} \end{vmatrix} + (0)\begin{vmatrix} \text{I.D.C} \end{vmatrix} + (0)\begin{vmatrix} \text{I.D.C} \end{vmatrix}$$

$$= (2)(5)\begin{vmatrix} 8 & 3 \\ 2 & 1 \end{vmatrix} = (10)(8-6) = 20$$

det can be used to solve system (Cramer's Rule)

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$
$$\longrightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Cramer's Rule:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

← det of A w/ 1st col replaced by $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

← det A

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

← det of A w/ 2nd col replaced by $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$x_1 + 3x_2 = 9$$
$$2x_1 + x_2 = 8$$

$$x_1 = \frac{\begin{vmatrix} 9 & 3 \\ 8 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-15}{-5} = 3$$

$$x_2 = \frac{\begin{vmatrix} 1 & 9 \\ 2 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-10}{-5} = 2$$

same idea for 3×3, 4×4, etc the transpose of A is a matrix w/ the rows and cols of A interchanged

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{its transpose} \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

notice $\det A = \det A^T$ (for all $n \times n$)

non zero

triangular matrix : $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ upper triangle

$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ lower triangle

non zero

det of triangular : product of main diagonal (for all $n \times n$)

a $\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = (1)(3) - (2)(0) = 3$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix}$$

row 3 expansion

$$6\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = (6)(1)(4)$$

how do row operations affect determinant?

→ any time two row are swapped, det changes sign

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \det A = -2$$

$$B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \rightarrow \det B = 2$$

→ multiplying one row by k $(k \neq 0)$ ~~multiplying~~ multiplies the det by the same k

$$A = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ 3 & 4 \end{bmatrix} \rightarrow \det A = \frac{4}{10} - \frac{3}{5} = -\frac{1}{5}$$

row 1 of A multiplied by 10

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \det B = -2 \quad (-\tfrac{1}{5} \cdot 10)$$

→ adding multiples of one row to another does NOT affect determinant

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \det A = -2$$

$$\xrightarrow{(-3)R_1 + R_2} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \quad \det A = -2 \quad \text{(same)}$$

example

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -3 & 2 \\ 5 & 1 & 9 \end{bmatrix}$$ no row/col w/ many zeros

reduce to triangular then find det by multiplying main diagonal or at least introduce zeros in a col or row then expand

do NOT affect det $\left\{ \begin{array}{l} (3)R_3 + R_2 \\ (2)R_3 + R_1 \end{array} \right. \longrightarrow \begin{bmatrix} 12 & 0 & 21 \\ 18 & 0 & 29 \\ 5 & 1 & 9 \end{bmatrix} = (-1) \begin{vmatrix} 12 & 21 \\ 18 & 29 \end{vmatrix} = 30$

good col to expand

because $\det A = \det A^T$ row and columns are interchangeable

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -3 & 2 \\ 5 & 1 & 9 \end{bmatrix}$$

$$\xrightarrow{C_2 + C_1} \begin{bmatrix} 0 & -2 & 3 \\ 0 & -3 & 2 \\ 6 & 1 & 9 \end{bmatrix}$$

good col to expand

looks like we did <u>column ops</u>

for the purpose of finding determinants, we can do <u>column operations</u> instead of row operations

$$A^T = \begin{bmatrix} 2 & 3 & 5 \\ -2 & -3 & 1 \\ 3 & 2 & 9 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_1} \begin{bmatrix} 0 & 0 & 6 \\ -2 & -3 & 1 \\ 3 & 2 & 9 \end{bmatrix}$$

good row to expand