3.6 Determinants (part 2)

transpose of A: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ rows $\leftrightarrow$ cols

note $\det A = \det A^T$

true for 3x3 and beyond, too

$$A = \begin{bmatrix} 3 & 0 & 3 \\ 3 & 4 & 3 \\ 0 & 5 & -2 \end{bmatrix} \qquad A^T = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 4 & 5 \\ 3 & 3 & -2 \end{bmatrix}$$

choose col 1

$$\det A = (3)\begin{vmatrix} 4 & 3 \\ 5 & -2 \end{vmatrix} - (3)\begin{vmatrix} 0 & 3 \\ 5 & -2 \end{vmatrix} = (3)(-23) - (3)(-15)$$

$$= -69 + 45 = -24$$

choose row 1

$\det A^T =$ same work as above $= \dots = -24$

triangular matrix :

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

~~lowe~~
upper triangular
(all #s below main diagonal are 0)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

lower triangular
(all #s above main diagonal are 0)

determinant of triangular matrix is the <u>product of the main diagonal numbers</u>

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix} \qquad \overset{\text{row 1}}{\det A} = (1)\begin{vmatrix} 3 & 0 \\ 5 & 6 \end{vmatrix} = (1)(3)(6) = 18$$

zeros are good for determinants

can we do row operations to introduce zeros (or make matrix triangular) and then find determinant ?

yes, but ...

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ det $A = -2$

row operation of swapping rows:

$\xrightarrow{swap(R_1,R_2)} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ determinant is 2

each swap of two rows switches the sign of the determinant

$A = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ 3 & 4 \end{bmatrix}$ $\det A = \frac{2}{5} - \frac{3}{5} = -\frac{1}{5}$

$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det B = -2 = 10 \cdot -\frac{1}{5} = 10 \det A$

(row 1 of A times 10)

multiplying one row by k ($k \neq 0$) changes the determinant by a factor of k

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$ det $A = -2$

same

$$\xrightarrow{(-3)R_1 + R_2} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$ ~~= (1)(~~ det is $(1)(-2) = -2$

upper triangular

$\boxed{\text{multiplying one row and add to another does NOT affect the determinant}}$

this is good because we use the operation to introduce zeros

example

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -3 & 2 \\ 5 & 1 & 9 \end{bmatrix}$$

let's try to introduce zeros or make it triangular

notice col 2 has the easiest numbers to use to introduce zeros (col 1/3 involve fractions)

these do NOT change the determinant

$$\xrightarrow[(3)R_3 + R_2]{(2)R_3 + R_1} \begin{bmatrix} 12^{+} & 0^{-} & 21^{+} \\ 18^{-} & 0^{+} & 29^{-} \\ 5^{+} & 1^{-} & 9^{+} \end{bmatrix}$$

now col 2 is good to expand on

$$\det A = -(1)\begin{vmatrix} 12 & 21 \\ 18 & 29 \end{vmatrix} = 30$$

example

$$A = \begin{bmatrix} 2 & 4 & -2 & 6 \\ 1 & 2 & 5 & 4 \\ 1 & 1 & 2 & 4 \\ 0 & 2 & -6 & 3 \end{bmatrix}$$

col 1 has nice #s so there is room to introduce more zeros

makes upper a 1 which is good pivot

$$\xrightarrow{\text{Swap}(R_1, R_3)} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 5 & 4 \\ 2 & 4 & -2 & 6 \\ 0 & 2 & -6 & 3 \end{bmatrix}$$

one swap → det flips sign

these do NOT change det

$$\xrightarrow[(-2)R_1 + R_3]{(-1)R_1 + R_2} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & -6 & -2 \\ 0 & 2 & -6 & 3 \end{bmatrix}$$

$$\xrightarrow[(-2)R_2+R_4]{(-2)R_2+R_3}\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -12 & -2 \\ 0 & 0 & -12 & 3 \end{bmatrix}$$

$$\xrightarrow{(-1)R_3+R_4}\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -12 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\det A = -(1)(1)(-12)(5) = 60$$

from swap earlier

$$\det A = \underbrace{\det A^T}$$

col operations $\longleftrightarrow$ row operations

row ops $\longleftrightarrow$ col ops

normally, we do <u>NOT</u> do column operations (they change solutions to the system)

but, for the purpose of finding det, we can do column ops

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\xrightarrow{(-2)C_1 + C_2} \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \qquad \xrightarrow{(-2)R_1 + R_2} \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$$

$\det A = -2$ $\qquad$ $\det A^T = -2$