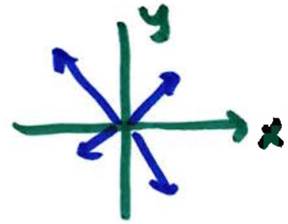
4.2 The Vector Space R^n and Subspaces

R^n : vector space containing vectors with n components

R^1 : real numbers

R^2 : vectors w/ 2 components



R^3 : x, y, z coordinate system

⋮

R^n : same idea

the reason we can operate with them (e.g. adding them) the same way we operate with numbers is because they are all in vector spaces vector space is a space that has the following 8 fundamental properties

1. $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ (e.g. in R^1, $1+2=2+1$)

2. $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ (e.g. in R^1, $1+(2+3)=(1+2)+3$)

3. there is a zero vector $\vec{0}$ such that $\vec{u}+\vec{0}=\vec{u}$

4. $\vec{u}+(-\vec{u}) = (-\vec{u})+\vec{u} = \vec{0}$ vector has its own additive inverse

5. $a(\vec{u}+\vec{v}) = a\vec{u}+a\vec{v}$

6. $(a+b)\vec{u} = a\vec{u}+b\vec{u}$

7. $a(b\vec{u}) = (ab)\vec{u}$

8. there is a 1 (one) such that $1\cdot\vec{u}=\vec{u}$

other things that not vectors in traditional sense can also be in vector spaces for example, M_2 : vector space containing all 2x2 matrices

"1" is the 2x2 identity

P_2 : all second degree polynomials

ax^2+bx+c

and lots of others

a lot of times we work with <u>subspaces</u> of known vector spaces what is a subspace?

it is a part of a vector space (has all 8 properties) such that it is closed under addition and scalar multiplication

closed under addition: sum of "vectors" in subspace remains in the same subspace

closed under scalar multiplication: scalar multiple remains in the same subspace

for example, R^2 is contained in R^3

two ~~ab~~ arbitrary R^2 vectors in R^2 remains in R^2

$\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\vec{v} = \begin{bmatrix} c \\ d \end{bmatrix}$ $\vec{u} + \vec{v} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$ still some R^2 vector

a, b, c, d are numbers

Scalar multiplication: $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\quad k\vec{u} = \begin{bmatrix} ka \\ kb \end{bmatrix}$

number

number

remains an R^2 vector

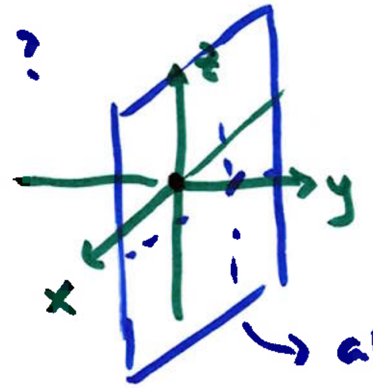
equivalently: linear combinations remain in the same space

→ implies the zero vector MUST be in a subspace

ANY subspace MUST have ALL of these:

1. closed under addition
2. closed under scalar multiplication
3. has zero vector

another example: is part of R^3 such that $y = 1$ a subspace?



all vectors in this plane

closed under addition?

$$\vec{u} = \begin{bmatrix} a \\ 1 \\ b \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} c \\ 1 \\ d \end{bmatrix}$$ a, b, c, d numbers

$$\vec{u} + \vec{v} = \begin{bmatrix} a+c \\ 2 \\ b+d \end{bmatrix}$$ is NOT a vector from that plane!

it left the plane!

it is NOT closed under addition

so that plane is NOT a subspace of R^3

an important subspace we often work with is the

null space of a matrix

→ all possible solutions to $A\vec{x} = \vec{0}$

(all possible $\vec{x}$ that get sent to 0)

example

$$\begin{aligned} x_1 - 4x_2 - 3x_3 - 7x_4 &= 0 \\ 2x_1 - x_2 + x_3 + 7x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 11x_4 &= 0 \end{aligned}$$

$$A\vec{x} = \vec{0}$$

$$\underbrace{\begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(null space of A contains ALL vectors the matrix transforms into $\vec{0}$)

$$\left[\begin{array}{ccccc} 1 & -4 & -3 & -7 & 0 \\ 2 & -1 & 1 & 7 & 0 \\ 1 & 2 & 3 & 11 & 0 \end{array}\right]$$

solve the system

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \to \cdots \to \left[\begin{array}{ccccc} \boxed{1} & 0 & 1 & 5 & 0 \\ 0 & \boxed{1} & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \leftarrow \text{zero row} \end{array}$$

no pivots

4 variables, 2 pivots $\to$ 2 free

x_3, x_4 have no pivots, so they are free

$x_3 = r$

$x_4 = t$

row 2: $\cdots \to x_2 = -r - 3t$

row 1: $\cdots \to x_1 = -r - 5t$

the vectors the matrix sent to $\vec{0}$ look like

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -r-5t \\ -r-3t \\ r \\ t \end{bmatrix}$$

$$= \underbrace{r\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$$

the matrix A sends ALL linear combos of $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ to $\vec{0}$

null space holds all these