4.3 Linear Combinations and Linear Independence

vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n$

they are linearly independent if and only if

$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$ implies ALL c_i are zero

if these vectors are columns of a matrix, then the vectors being linearly indp if ~~they~~ there are as many pivots as there are columns

otherwise, the equation $\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}}_{\substack{\text{matrix w/} \\ \vec{v}_i \text{ as columns}}} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

will have more than one solution, but linearly indp means a unique solution $c_1 = c_2 = \cdots = c_n = 0$ (trivial solution) for example, $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{\text{reduction}} \begin{bmatrix} \boxed{1} & 2 \\ 0 & \boxed{-2} \end{bmatrix}$$ two pivots

that means $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix} \rightarrow \begin{array}{ccc} c_1 & c_2 & \\ \hline \boxed{1} & 2 & 0 \\ 0 & \boxed{-2} & 0 \end{array}$$

both c_1 and c_2 are zero and only zero (no free variables)

so, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ are linearly indp

given vectors, if want to find if they are linearly indp, put in as cols of matrix, then count pivots for example, $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v_4} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 3 \end{bmatrix} \to \cdots \to \begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 2 \end{bmatrix}$$

3 pivots, 4 vectors $\to$ $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + c_4\vec{v_2} = \vec{0}$

has free variables $\to$ multiple solutions

So, these vectors are NOT linearly indp

$\to$ this set of FOUR vector are dependent

(one can be expressed as linear combo of others)

$\to$ BUT, a subset of these four may be indp

(e.g. $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$ ARE indp)

notice the max # of pivots is the # of rows $\to$ # of components in each vector

$\to$ if there are more vectors than # of componets, the vectors are automatically linearly dependent

BUT, if ~~they~~ there are as many vectors as components or fewer vectors, then the vectors may or may not be indp.

for example, $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

two R^2 vectors (same # components and vectors)

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 \\ 0 & 0 \end{bmatrix}$$

one pivot so NOT indp

if $\begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix}$ is square, then indp $\rightarrow$ determinant is zero

<u>example</u> $\vec{v}_1 = \begin{bmatrix} -1 \\ -17 \\ -3 \\ 9 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 14 \\ 7 \\ 2 \\ -2 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 15 \\ 5 \\ 1 \\ -2 \end{bmatrix}$

3 $\mathbb{R}^4$ vectors → may or may not be indp

$$\begin{bmatrix} -1 & 14 & 15 \\ -17 & 7 & 5 \\ -3 & 2 & +1 \\ 9 & -2 & -2 \end{bmatrix} \to \cdots \to \begin{bmatrix} \boxed{-1} & 14 & 15 \\ 0 & \boxed{-231} & -250 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

3 pivots

row of zeros → normally means free variables

we assign free variables to variables WITHOUT pivots in column

here, every column has pivot, so despite the zero row, there are <u>NO</u> free variables

3 pivots, 3 vectors → they ARE linearly indp.

next, the <u>span</u> of a set of vectors

Span: the "things" you can make with the given vectors

often we talk about the <u>space</u> spanned by the vectors

for example, $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

it is clear we can make <u>every</u> possible R^2 vector using linear combos of these

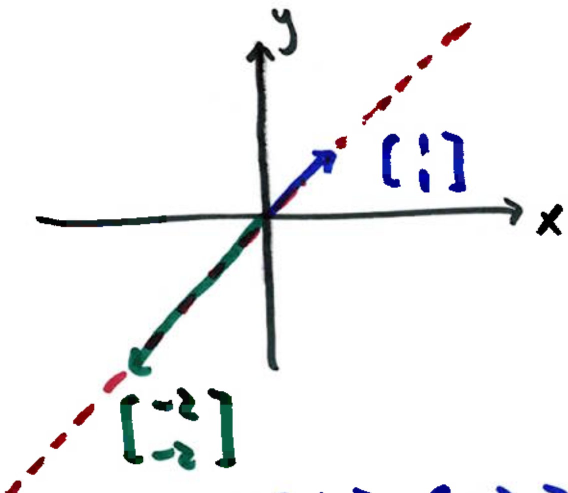
$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so, we say that <u>$\vec{i}$ and $\vec{j}$ span R^2</u>

likewise, $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ span R^3

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = R^3$$

what is span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right\}$?



span : linear combos

$$a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -2 \\ -2 \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

because $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$ is dep multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

↳ span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right\} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

what about span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$?

it is still R^2; $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not needed but the set still spans R^2

we can have more vectors than we need in a spanning set the minimum we need appears to be the number of components in each vector

R^n : needs n vector at least

we need <u>exactly</u> n linearly independent vectors

$$R^2: \left\{ \underbrace{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}}_{\text{2 indp}}, \underbrace{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}, \begin{bmatrix}\pi\\e\end{bmatrix}}_{\text{extras}} \right\}$$

also ok: $R^2: \left\{ \underbrace{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}}_{\text{indp}} \right\}$