4.4 Bases and Dimension of Vector Space

R^2 is spanned by $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \{ \vec{i}, \vec{j} \}$

because every R^2 vector is a linear combo of $\vec{i}$ and $\vec{j}$

$$\begin{bmatrix} 5 \\ 7 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 5\vec{i} + 7\vec{j}$$

spanning set : enough vectors to build a vector space

R^2 is also spanned by $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

(we can simply ignore the extra one)

R^2 is not spanned by just $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ or just $\vec{i}$

(missing y component)

R^2 is not spanned by $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$ (missing y)

but we can span R^2 using $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

for R^2, we need a minimum of Two linear indp vectors in the spanning set

Similarly, in R^3 : $\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = R^3$

$$\begin{bmatrix}1\\2\\3\end{bmatrix} = 1\vec{i} + 2\vec{j} + 3\vec{k}$$

also spanned by $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}\pi\\e\\10\end{bmatrix}\right\}$

but not by $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

but not by $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$

this is ok: $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\\pi\end{bmatrix}\right\}$

for R^n, we need a minimum of n linear indp vectors to span.

the smallest spanning set (n linear indp vectors) is called a basis (the vectors in the set are called basis vectors or bases) for a given vector space, the basis is NOT unique

for example, in R^2, the standard basis is $\vec{i}$, $\vec{j}$

$$\begin{bmatrix} 5 \\ 7 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

but $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}$ is also a basis for R^2

$$\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \frac{57}{10} \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \frac{7}{5} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

also ok : $\{ 3\vec{i}, -10\vec{j} \}$

R^3: standard basis $\vec{i}$, $\vec{j}$, $\vec{k}$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\vec{i} + 2\vec{j} + 3\vec{k}$$

also a basis: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$$

the <u>dimension</u> of a vector space is <u>the number of vectors in its basis</u>

R^2 : needs Two basis vectors → Two-dimensional

R^n : " n " " → n-dim

M_2 : all possible 2×2 matrices

one easy basis: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

so M_2 is a 4-D vector space

P_2 : all possible 2nd-deg polynomials

$ax^2 + bx + c$

basis: $\{1, x, x^2\}$ → 3-D vector space

<u>example</u> The subspace of R^3 given by the plane $3x+2y+z=0$

all vectors in this subspace is $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -3x-2y \end{bmatrix} = \underbrace{x\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}}$$

linear combos of $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ span (and is a basis for) this subspace

one possible basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$ 2-D subspace

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}y - \frac{1}{3}z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}$$

another basis: $\left\{ \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$

note $\left\{ \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\}$ is also a basis

-3 times
first vector
of last set

3 times
2nd vector

and many other possible sets

<u>example</u> solution space of

$$x_1 - 2x_2 - 5x_3 = 0$$
$$2x_1 - 3x_2 - 13x_3 = 0$$

solution space: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ that satisfies the eqs above

it is also the <u>null space</u> of the <u>coefficient matrix</u>

$$\begin{bmatrix} 1 & -2 & -5 \\ 2 & -3 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

what does this space look like?

solve the system:

$$\begin{bmatrix} 1 & -2 & -5 & 0 \\ 2 & -3 & -13 & 0 \end{bmatrix}$$

$$\to \cdots \to \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \begin{bmatrix} \boxed{1} & -2 & -5 & 0 \\ 0 & \boxed{1} & -3 & 0 \end{bmatrix} \end{array}$$

3 variables, 2 pivots

$\to$ one free variable

choose x_3 to be free

(no pivot in its col)

$x_3 = r$

row 2: $x_2 = 3r$

row 1: $\cdots$ $x_1 = 11r$

solution: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11r \\ 3r \\ r \end{bmatrix} = r \begin{bmatrix} 11 \\ 3 \\ 1 \end{bmatrix}$

null space / solution space contains all possible scalar multiples of $\begin{bmatrix} 11 \\ 3 \\ 1 \end{bmatrix}$

space is spanned by $\begin{bmatrix} 11 \\ 3 \\ 1 \end{bmatrix}$ which is also a basis

so, the null space in this example is ONE-dimensional

here, the intersection of the two planes is a line