4.4 Bases and Dimension of a Vector Space

Span: vectors needed to reach an entire vector space

for example, $\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \text{span}\{\vec{i}, \vec{j}\} = R^2$

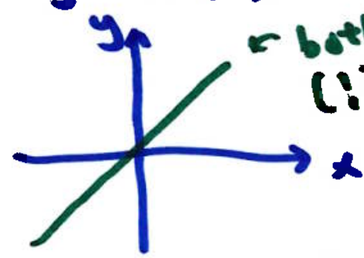
any R^2 vector can be expressed as a linear combo of $\vec{i}$ and $\vec{j}$

e.g. $\begin{bmatrix}5\\7\end{bmatrix} = 5\begin{bmatrix}1\\0\end{bmatrix} + 7\begin{bmatrix}0\\1\end{bmatrix} = 5\vec{i} + 7\vec{j}$

R^2 is also spanned by $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}$ (more than enough for R^2)

but R^2 is NOT spanned by just $\vec{i} = \begin{bmatrix}1\\0\end{bmatrix}$ (no y-component)

also NOT spanned by $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\2\end{bmatrix}$ (not entire R^2)



← both $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\2\end{bmatrix}$ are on this line try R^3: spanned by $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

also spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ more than enough

but NOT spanned by $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (no z)

also NOT spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ (no z)

So, for R^n, we need at least n vectors, but n may not be enough

→ for n vectors to span R^n, they need to be linearly independent

minimum: n linearly indp vectors for R^n

we can use more than n to span, as long as n of them are linearly indp

so, need n linearly indp in our set (w/ equal to or more than n vectors) the <u>minimum spanning set</u> for a vector space is called a <u>basis set</u> (or just simply basis)

plural : bases

the basis set for a vector space is <u>NOT</u> unique

R^2: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis

notice $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}$ also span (and is basis) for R^2

$$\begin{bmatrix} 10 \\ -7 \end{bmatrix} = 10 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 7 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 10 \\ -7 \end{bmatrix} = \frac{57}{10} \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \frac{7}{5} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

ANY Two linearly indp is ok

$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are convenient (standard basis for R^2)

R^3: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis set

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix} \right\}$ is also a basis

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\vec{i} + 2\vec{j} + 3\vec{k}$$

$$= -\frac{1}{2}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$$

basis is <u>NOT</u> unique

but there is a unique linear combo for any vector in a given basis set.

true for all R^n

and ANY vector space (e.g. M_2 : all 2x2 matrices) dimension of a vector space is the number of vectors in its basis set

R^2 : need Two basis vectors → Two dimensional vector space

R^n : " N " " → N " " "

M_2 : need FOUR basis "vectors" → FOUR-D vector space

example Find the dimension and a basis for the subspace of R^3 given by the plane $3x+2y+z=0$

ANY vector in ~~sp~~ subspace $3x+2y+z=0$ looks like

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -3x-2y \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}y - \frac{1}{3}z \\ y \\ z \end{bmatrix}$$

$$x\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

→ ALL vectors in subspace
every vector is a linear combo
of those Two

$\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ span the subspace

and linearly indp $\to$ they form a basis

so, the subspace of R^3 is a <u>Two-dimensional subspace</u>

using $\begin{bmatrix} -\frac{2}{3}y - \frac{1}{3}z \\ y \\ z \end{bmatrix}$ we can get another basis

$$= y\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

new basis: $\left\{ \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$

another set: $\left\{ \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\}$ (above multiplied by 3)

any nonzero multiple of a basis is another basis example Find a basis for the solution space

to $x_1 - 2x_2 - 5x_3 = 0$

$2x_1 - 3x_2 - 13x_3 = 0$

find ALL $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that

$$\underbrace{\begin{bmatrix} 1 & -2 & -5 \\ 2 & -3 & -13 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{0}}$$

$A\vec{x} = \vec{0}$

→ all $\vec{x}$ that A sends to $\vec{0}$

null space of A

$$\begin{bmatrix} 1 & -2 & -5 & 0 \\ 2 & -3 & -13 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \left[\begin{matrix} \boxed{1} & -2 & -5 & 0 \\ 0 & \boxed{1} & -3 & 0 \end{matrix}\right] \end{array}$$

3 variables, 2 pivots, x_3 w/o pivot $\rightarrow$ x_3 is free

$x_3 = r$

row 2: $x_2 = 3r$

row 1: $x_1 = \cdots = 11r$

all vectors that live in null space of A

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11r \\ 3r \\ r \end{bmatrix} = r \begin{bmatrix} 11 \\ 3 \\ 1 \end{bmatrix} \rightarrow \text{spanned by } \begin{bmatrix} 11 \\ 3 \\ 1 \end{bmatrix}$$

$\rightarrow$ the null space is ONE-dimensional

this null space is a line (intersection of two planes)