4.5 Row and Column Spaces

given a matrix, all possible linear combos form the row space
(space spanned by its rows)

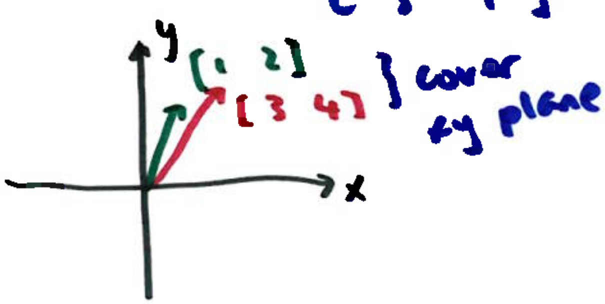
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \mathrm{Row}(A) = \mathrm{span}\left\{ [1 \;\; 2], [3 \;\; 4] \right\}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathrm{Row}(B) = \mathrm{span}\left\{ [1 \;\; 0], [0 \;\; 0] \right\}$$

$$= \mathrm{span}\left\{ [1 \;\; 0] \right\}$$

what about a basis (minimum spanning set) of row space?

for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $[1 \;\; 2]$ and $[3 \;\; 4]$ are linearly indp so form a basis for Row(A) which is R^2



so another basis is $\left\{ [15 \;\; 0], [0 \;\; \pi] \right\}$

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 4 & 3 \\ 2 & 2 & 5 \end{bmatrix}$$

Row(A)?
find a basis

Row(A) = span $\{[1 \ -2 \ 2], [1 \ 4 \ 3], [2 \ 2 \ 5]\}$

but a basis? knowing a basis helps us see if the Row(A) is, for example, $\mathbb{R}^3$

we need to find linearly indp vectors that span the same space (Row(A))

Gaussian elimination produces <u>row equivalent</u> matrices and <u>preserves</u> row space

for example, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = B$

NOT equal
but are row equivalent
<u>same</u> row space

Row(B) = span $\{[1 \ 2], [0 \ -2]\}$ = Row(A) back to $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 4 & 3 \\ 2 & 2 & 5 \end{bmatrix}$

$$\to \cdots \to \begin{bmatrix} 1 & -2 & 2 \\ 0 & 6 & 1 \\ 0 & 0 & 0 \end{bmatrix} = B \qquad \text{Row}(A) = \text{Row}(B)$$

$$\text{Row}(B) = \text{span}\Big\{ \underbrace{[1 \ \ -2 \ \ 2], [0 \ \ 6 \ \ 1]}_{\text{linearly indp}} \Big\}$$

$$= \text{Row}(A)$$

so, a basis for Row(A) is $\{[1 \ \ -2 \ \ 2], [0 \ \ 6 \ \ 1]\}$

To find a basis for row space, perform row reduction and identify the <u>pivot rows</u> (rows w/ pivots) and those pivots form a basis for row space

\# of pivots → <u>rank</u> of the matrix

matrix A above is rank two

Column space: span of columns

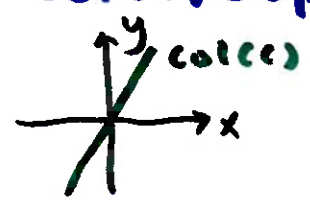
all possible linear combos of columns

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 8 & 8 \end{bmatrix} \qquad \mathrm{Col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} \right\}$$

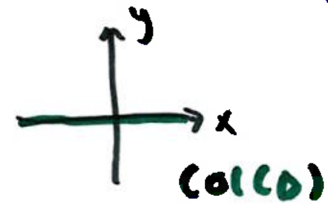
can we find a basis?

row ops preserve row space but does NOT, in general, preserve column space

for example, $C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ $\quad \mathrm{col}(C) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$



$\rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = D \quad \mathrm{col}(D) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$



even though row ops change column space, ~~the~~ the ~~lineart~~ linear independency of column is preserved

for example, $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ clearly, cols 1, 2 are indp

$$\rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix}$$

column 1, 2 pivot columns so are indp.

whatever columns that were indp remain indp after row reduction

use the reduced matrix to find the columns that were indp in the original matrix back to $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 8 & 8 \end{bmatrix}$ $\quad Col(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} \right\}$

reduce: $\longrightarrow \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 1 \\ 0 \end{matrix}} & \begin{matrix} 4 \\ -2 \\ 0 \end{matrix} \end{bmatrix}$

pivot columns: these do NOT for a basis for Col(A)

but this tells us that columns 1 and 2 in A are indp

so, since cols 1, 2 in A are indp, they form a basis for Col(A)

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix} \right\}$$

<u>example</u> $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 5 & -9 & 10 \\ 1 & 6 & -29 & -11 \end{bmatrix}$ find basis for Row(A) and Col(A)

$$\to \cdots \to \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{2} & -12 & 7 \\ 0 & 0 & 0 & \boxed{-59} \end{bmatrix}$$

pivot rows: all 3, so they form a basis for Row(A)

$$\{ [1 \ 1 \ 1 \ 1], [0 \ 2 \ -12 \ 7], [0 \ 0 \ 0 \ -59] \}$$

pivot colums: 1, 2, 4, so we pick columns 1, 2, 4 in the original matrix to form a basis for Col(A)

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ -11 \end{bmatrix} \right\}$$

NOT a basis: for Col(A) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -59 \end{bmatrix} \right\}$

given a set of vectors, if we want to know which are linearly indp. put them in as columns of a matrix and identify pivot columns which tells us which vectors are indp

(see previous example pretending the cols are the vectors)

another way to find basis for column space:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -9 & 17 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & -9 \\ 3 & 17 \end{bmatrix}$$

$$\mathrm{Row}(A^T) = \mathrm{Col}(A)$$

reduce A^T: $\rightarrow \cdots \rightarrow \begin{bmatrix} \boxed{1} & 4 \\ 0 & \boxed{-17} \\ 0 & 0 \end{bmatrix}$

pivot rows: $[1 \quad 4], [0 \quad -17]$

$\rightarrow$ basis for $\mathrm{Row}(A^T)$

so, a basis for $\mathrm{Col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -17 \end{bmatrix} \right\}$