4.5 Row and Column Spaces

given matrix A, the vector space spanned by its rows is called the row space of A → $Row(A)$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$Row(A)$ spanned by $\{[1 \ \ 2], [0 \ \ 1]\}$

notice for this A, the rows are linearly indp so a basis for $Row(A)$ is $\{[1 \ \ 2], [0 \ \ 1]\}$

(another one: $\{[1 \ \ 0], [0 \ \ 1]\}$

so, the $Row(A)$ here is R^2 → two-dim row space

try this one:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$Row(B) = span\{[1 \ \ 0], [0, 0]\}$

basis: $\{[1 \ \ 0]\}$ → one-dim row space

$$C = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 4 & 3 \\ 2 & 2 & 5 \end{bmatrix}$$ find basis for row space

row space is spanned by $\{[1\ -2\ 2], [1\ 4\ 3], [2\ 2\ 5]\}$

this set may not be a basis → needs to be linearly indp

so, we need to see which vectors ~~are~~ form a linearly indp set → a basis

Gaussian elimination uncovers the linear indp between the rows

$$C \to \cdots \to \begin{bmatrix} 1 & -2 & 2 \\ 0 & 6 & 1 \\ 0 & 0 & 0 \end{bmatrix} = D$$

C is <u>NOT</u> equal to D

but C and D are <u>row equivalent</u>

(row ops produce row eq. matrices)

row eq. matrices have the same row space

$\rightarrow$ row ops. preserve row space

$$C = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 4 & 3 \\ 2 & 2 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 6 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

SAME row space

$$\text{Row}(C) = \text{span}\left\{ \begin{bmatrix} 1 & -2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 5 \end{bmatrix} \right\}$$

$$= \text{Row}(D) = \text{span}\underbrace{\left\{ \begin{bmatrix} 1 & -2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 6 & 1 \end{bmatrix} \right\}}_{\text{linearly indp so is a basis for Row(D) AND Row(C)}}$$

$$\boxed{\text{the pivot rows of a matrix form a basis for the row space of the matrix}}$$

the rank of a matrix is the number of pivots (which is the dimension of its row space)

so the C matrix above is a rank TWO matrix the vector space spanned by a matrix's <u>columns</u> is its <u>column space</u> $\rightarrow$ Col(A)

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 8 & 8 \end{bmatrix} \quad \text{Col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} \right\}$$

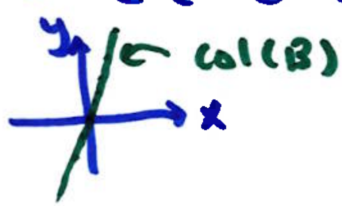
is this a basis?

if not, can we find a basis?

Still need to find the vectors among them that are linearly indp.

$\boxed{\text{row operations, in general, do } \underline{\text{NOT}} \text{ preserve column space}}$

for example, $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ $\text{col}(B) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$



reduce B: $\rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = C$

$\text{col}(C) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Even though row ops do NOT preserve column space,

row ops do preserve the linear independency between columns

$$A \to \cdots \to B$$

linear indp columns correspond to the same columns in A

$$\boxed{\begin{array}{l}\text{to find a basis for col(A), we reduce A and identify the pivot columns in the reduce matrix}\\ \text{the corresponding } \cancel{\text{those}} \text{ columns in A are linear indp so form a basis for col(A)}\end{array}}$$

example

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 8 & 8 \end{bmatrix}$$

Find a basis for Row(A) and Col(A)

$$\xrightarrow{\text{row ops}} \cdots \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = B$$

Since row ops preserve row space, pivot rows of B form a basis for Row(B) AND Row(A)

$$\boxed{\text{so, a basis for Row(A) is } \{[1 \;\; 2 \;\; 4], [0 \;\; 1 \;\; -2]\}}$$

Col(A), in general, $\neq$ Col(B), so we can't use pivot columns of B to form a basis for Col(A)

however, the corresponding columns in A (where B has pivot columns) ARE linearly indp

cols 1 and 2 in B are indp, so cols 1 and 2 of A are also linearly indp so form a basis

$$\boxed{\text{a basis for Col(A) is } \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix} \right\}}$$

alternatively, to find basis for Col(A) we can find a basis for Row(A^T)

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 8 & 8 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 8 \\ 4 & 2 & 8 \end{bmatrix}$$

find basis for row space

$$\to \cdots \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix}$$

$$\to \begin{bmatrix} \boxed{1 \quad 1 \quad 3} \\ 0 \quad \boxed{1 \quad 2} \\ 0 \quad 0 \quad 0 \end{bmatrix}$$

so, a basis for Row(A^T) is $\{[1 \;\; 1 \;\; 3], [0 \;\; 1 \;\; 2]\}$

and a basis for Col(A) is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

<u>example</u> $\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix}$

are they linearly indp?

if not, which ones are?

one way to find the answers is to put them as columns of a matrix and find a basis for the column space

$$A = \begin{bmatrix} 3 & 2 & 4 & 1 \\ 2 & 1 & 3 & 2 \\ 2 & 2 & 2 & 1 \\ 2 & 1 & 3 & 4 \end{bmatrix}$$ find basis for Col(A)

$$\rightarrow \cdots \rightarrow \begin{bmatrix} \boxed{3} & 2 & 4 & 1 \\ 0 & \boxed{-1} & 1 & 4 \\ 0 & 0 & 0 & \boxed{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivots in cols 1, 2, 4

cols 1, 2, 4 of A (not reduced) are linearly indp

so, $\vec{v}_1, \vec{v}_2, \vec{v}_4$ are linearly indp.

matrix A : col(A)

Row(A)

null(A)

one relationship : Row(A) $\perp$ null(A)

$\uparrow$ perpendicular

every vector in Row(A) $\perp$ every vector in null(A)

why?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -9 & 17 \end{bmatrix} \quad \text{Row}(A) = \text{span}\{[1\ 2\ 3], [4\ -9\ 17]\}$$

every vector in Row(A) is linear combo of the rows

null(A) : solution to $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & -9 & 17 & 0 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{-17} & 5 & 0 \end{bmatrix}$$

pivots $\quad \uparrow x_3$ free $\quad x_3 = r$

solve: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \cdots = r \begin{bmatrix} -61/17 \\ 5/17 \\ 1 \end{bmatrix}$

$$\text{null}(A) = \text{span}\left\{ \begin{bmatrix} -61/17 \\ 5/17 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -61/17\, r \\ 5/17\, r \\ r \end{bmatrix} = -\frac{61}{17}r + \frac{10}{17}r + \frac{51}{17}r = 0$$

same w/ row 2 and any linear combo of rows

dot product = 0 → perpendicular