5.1 Intro to 2nd-Order Eqs

Linear 2nd-order: $y'' + p(x)y' + g(x)y = f(x)$

cannot contain y
(else nonlinear)

first, let's investigate homogeneous equation

↳ right side $= 0$

$f(x) = 0$

$$y'' + p(x)y' + g(x)y = 0$$

to understand the structure of the solution, let's look at a very simple one: $y'' = 0$

$y'' = 0$

integrate: $y' = c_1$

integrate: $y = c_1 x + c_2$

this shows that the solution is a linear combo of

the functions $\underbrace{y_1 = x \;,\; y_2 = 1}$

solution is sum of scalar multiples of each

this means the two functions (which are linearly independent)

span the solution space

in fact, the two functions form a basis for the solution space

$$\boxed{\begin{array}{l}\text{general feature: solution to } y'' + p(x)y' + g(x)y = 0 \\ \text{is } y = c_1 y_1 + c_2 y_2 \;,\; y_1, y_2 \text{ span and form a basis for solution space}\end{array}}$$

$\boxed{\begin{array}{l}\text{linear independence of functions: if } y_1 \text{ and } y_2 \text{ are} \\ \quad \text{linearly indp on an interval of } x \\ \quad \text{then on that interval } c_1 y_1 + c_2 y_2 = 0 \text{ is possible} \\ \quad \text{if and only if } c_1 = c_2 = 0\end{array}}$

if y_1, y_2 are linearly indp on an interval

then $c_1 y_1 + c_2 y_2 = 0$

and $c_1 y_1' + c_2 y_2' = 0$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

unique solution of $c_1 = c_2 = 0$ if and only if

$$\boxed{\underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}_{\text{Wronskian of } y_1,\, y_2} \neq 0}$$

solution of $y'' + py' + qy = 0$ means the function

y that satisfies the eq.

(for 2nd order, there are TWO)

$$y'' - y = 0 \longrightarrow y'' = y$$

y is a function that's equal to

its 2nd derivative

$$y_1 = e^x \qquad y_2 = e^{-x}$$

check: $\left.\begin{array}{l} y_1' = e^x \\ y_2'' = e^x \end{array}\right\}$ sub into $y'' - y = 0$

$$e^x - e^x = 0 \quad \checkmark$$

$\left.\begin{array}{l} y_2 = e^{-x} \\ y_2' = -e^{-x} \\ y_2'' = e^{-x} \end{array}\right\}$ sub into $y'' - y = 0$

$$e^{-x} - e^{-x} = 0 \quad \checkmark$$

both $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions to $y'' - y = 0$

and so are all linear combos: $\underbrace{y = c_1 e^x + c_2 e^{-x}}_{\text{general solution}}$

$y_1 = e^x$ and $y_2 = e^{-x}$ are linearly indp because their Wronskian is never zero

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2 \neq 0$$

we guessed and checked to solve $y'' - y = 0$

a more systematic way to solve $y'' + py' + qy = 0$

is to use the characteristic polynomial or characteristic equation

$$y'' + p(x)y' + g(x)y = 0$$

for constant coefficient equation, the solutions are ALWAYS in the form $y = e^{rx}$ because e^{rx} has derivatives that are constant multiples of itself

$y'' + ay' + by = 0$ is ALWAYS solved by $y = e^{rx}$ r is const

$$\left\{\begin{aligned} y &= e^{rx} \\ y' &= re^{rx} \\ y'' &= r^2 e^{rx} \end{aligned}\right.$$

sub in

$$r^2 e^{rx} + are^{rx} + be^{rx} = 0$$

$(r^2 + ar + b)e^{rx} = 0$ $e^{rx} \neq 0$ for any r

so, $\boxed{r^2 + ar + b = 0}$ Characteristic Equation

Solve for r: two solution values r_1, r_2

solutions of diff. eq: $y_1 = e^{r_1 x}$ $y_2 = e^{r_2 x}$

$$\boxed{\text{general solution: } y = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$$

example

$$1y'' + 3y' + 2y = 0$$

constant coefficient

solution: $y = e^{rx}$

$$y' = re^{rx}$$

$$y'' = r^2 e^{rx}$$

Sub into eq. above

$$r^2 e^{rx} + 3re^{rx} + 2e^{rx} = 0$$

$$e^{rx}(r^2 + 3r + 2) = 0$$

$$\boxed{1r^2 + 3r + 2 = 0}$$ characteristic eq.

notice coeffs match the original diff. eq.

solve for r: $r^2 + 3r + 2 = 0$

$$(r+2)(r+1) = 0$$

$$r_1 = -2 \quad r_2 = -1$$

the two linearly indp solutions: $y_1 = e^{r_1 x} = e^{-2x}$

$y_2 = e^{r_2 x} = e^{-x}$

general solution is linear combo of them:

$$\boxed{y = c_1 e^{-2x} + c_2 e^{-x}}$$

to find these constants, we need TWO initial conditions

typically: $y(x_0) = y_0$ and $y'(x_0) = y_0'$

for example, suppose $y(0) = 1$ and $y'(0) = 0$

gen. solution: $y = c_1 e^{-2x} + c_2 e^{-x}$

sub in $y(0) = 1$

$c_1 e^0 + c_2 e^0 = 1$

$c_1 + c_2 = 1 \quad - \text{①}$

to use $y'(0) = 0$ we need y'

$$y' = -2c_1 e^{-2x} - c_2 e^{-x}$$

sub in $y'(0) = 0$

$$-2c_1 e^0 - c_2 e^0 = 0$$

$$-2c_1 - c_2 = 0 \quad - \text{②}$$

solve ①, ②

$$c_1 + c_2 = 1$$

$$-2c_1 - c_2 = 0$$

solve this system (Gaussian elimination)

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$c_2 = 2$$

$$c_1 + c_2 = 1$$

$$c_1 = 1 - 2 = -1$$

solution: $y = c_1 e^{-2x} + c_2 e^{-x}$

$$y = -e^{-2x} + 2e^{-x}$$

example $4y'' + 4y' + 1y = 0$

characteristic eq has same coefficients

$4r^2 + 4r + 1 = 0$ (if you forget, sub $y = e^{rx}$ into eq.)

$(2r+1)(2r+1) = 0$

$r_1 = -1/2 \quad r_2 = -1/2$ repeated roots

if roots are repeated $y_1 = e^{r_1 x}$ as usual

$y_2 = x e^{r_1 x}$ mult. by x

here, $y_1 = e^{-1/2 x}$

$y_2 = x e^{-1/2 x}$

general solution: $\boxed{y = c_1 e^{-1/2 x} + c_2 x e^{-1/2 x}}$