5.1 Intro to 2nd-Order Eqs

$$y'' + p y' + q y = f$$

if the functions p, q, f are functions of x (indp. variable)

$$y'' + p(x) y' + q(x) y = f(x)$$

no y

then the eq. is linear (else it is nonlinear)

to understand the basics of 2nd-order, let's start with homogeneous, constant-coefficient eq.

right side = 0

$f = 0$

p, q are constants

$$\rightarrow \boxed{y'' + a y' + b y = 0}$$

to understand the structure of the solution, let's solve a very easy one: $y'' = 0$

solution 1: any constant $\rightarrow$ constant multiple of 1

solution 2: any multiple of x

notice ~~the~~ any linear combo of 1 and x also solves $y'' = 0$

$$y = c_1 \cdot 1 + c_2 \cdot x$$
$$y' = c_2$$
$$y'' = 0$$

this means the solution space is spanned by the functions $y_1 = 1$ and $y_2 = x$

and in fact is a basis

(because 1 and x are clearly linear indp.)

linear independence of functions:

two functions y_1 and y_2 are linearly indp

if $c_1 y_1 + c_2 y_2 = 0$ on an interval of x

means $c_1 = c_2 = 0$ only

for example, $y_1 = 1$ and $y_2 = x$

they are linearly indp on all intervals

because $c_1 \cdot 1 + c_2 \cdot x = 0$ if any only if $c_1 = c_2 = 0$

for more general functions (e.g. e^x, $\cos x$, etc)

testing linearly indp. is a bit more difficult

than with 1 and x

notice if two functions y_1 and y_2 are linearly

indp. on some interval of

then $c_1 y_1 + c_2 y_2 = 0$ if and only if $c_1 = c_2 = 0$

its derivative: $c_1 y_1' + c_2 y_2' = 0$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we know the solution is unique if and only if

$$\underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}_{\text{Wronskian of } y_1 \text{ and } y_2} \neq 0$$

The general features of a homogeneous constant-coeff 2nd eq.

→ two linear ~~ly~~ independant functions spanning solution space

→ two functions are indp so their Wronskian is nonzero

How to solve $y'' + ay' + by = 0$?

→ this suggests y and its derivatives are constant multiples of one another

exponential → $y = e^{rx}$ r: constant

$y' = re^{rx}$ constant multiple of itself

$y'' = r^2 e^{rx}$ "   "

now let's plug $y = e^{rx}$ and y' and y'' into

$$y'' + ay' + by = 0$$

$$r^2 e^{rx} + are^{rx} + be^{rx} = 0$$

$$e^{rx}(r^2 + ar + b) = 0$$

since $e^{rx} \neq 0$ for any r or x

$$\boxed{r^2 + ar + b = 0}$$ Characteristic Equation

it is quadratic : two solutions r_1 and r_2

so, solutions to $y'' + ay' + by = 0$ are

$y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$ and ANY linear combo of them

so, the <u>general solution</u> is

$$\boxed{y = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$$

<u>example</u>

$y'' - y = 0 \rightarrow 1 \cdot y'' + 0 y' + 1 \cdot y = 0$

$y = e^{rx}$ $y' = re^{rx}$ $y'' = r^2 e^{rx}$

$r^2 e^{rx} - e^{rx} = 0$

$e^{rx}(r^2 - 1) = 0$

$1 \cdot r^2 - 1 = 0$ characteristic eq.

matches coeff of y''

$0 \cdot r$

coeff matches y'

matches coeff of y

$r^2 - 1 = 0 \rightarrow r_1 = -1$ and $r_2 = 1$

so, $y_1 = e^{r_1 x} = e^{-x}$ is one solution

$y_2 = e^{r_2 x} = e^{x}$ is another

general solution: any linear combo of them

$$\boxed{y = c_1 y_1 + c_2 y_2 = c_1 e^{-x} + c_2 e^{x}}$$

$y_1 = e^{-x}$ and $y_2 = e^{x}$ are linearly indp

Wronskian: $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix}$

$= 1 - (-1) = 2 \neq 0$ on any interval of x

($W \neq 0 \rightarrow$ linear indp.)

<u>example</u> $1 \cdot y'' + 3y' + 2y = 0$

coeff of r^2 coeff of r constant

characteristic eq: $1 \cdot r^2 + 3r + 2 = 0$

(always obtainable by subbing $y = e^{rx}$ into diff. eq.)

$$r^2 + 3r + 2 = 0$$

$$(r+2)(r+1) = 0$$

$$r = -2, -1$$

general solution: $\boxed{y = c_1 e^{-2x} + c_2 e^{-x}}$

to find c_1 and c_2, we need TWO initial conditions

often $y(x_0) = y_0$ but can also be two

$y'(x_0) = y_0'$ y ~~ys~~ or two y'

for example, if for this eg.

$y(0)=1$ and $y'(0)=0$

$$y=c_1e^{-2x}+c_2e^{-x}$$

$y(0)=1 \rightarrow \quad 1=c_1+c_2 \quad -\ 1$

to use $y'(0)=0$, we need y'

$$y'=-2c_1e^{-2x}-c_2e^{-x}$$

$y'(0)=0 \rightarrow \quad 0=-2c_1-c_2 \quad -\ 2$

solve 1, 2

$1+2 \quad 1=-c_1 \rightarrow c_1=-1$

$1: \quad c_2=1-c_1 \rightarrow c_2=2$

so, the solution is

$$\boxed{y=-e^{-2x}+2e^{-x}}$$

if the roots of the characteristic eq. are the same

$$r^2 + ar + by = 0 \rightarrow r_1 = r_2$$

then obviously $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are the same and therefore NOT linearly indp.

fix: $\boxed{\begin{array}{ll} y_1 = e^{r_1 x} & \text{(usual)} \\ y_2 = x e^{r_2 x} & \text{(multiply by } x\text{)} \end{array}}$ repeated r value only

(this guarantees non-zero Wronskian and therefore linear indp) example $4y'' + 4y' + y = 0$

characteristic eq: $4r^2 + 4r + 1 = 0$

$(2r+1)(2r+1) = 0$

$r = -\frac{1}{2}, -\frac{1}{2}$

solutions: $y_1 = e^{-\frac{1}{2}x}$

$y_2 = \underline{\underline{x}}\, e^{-\frac{1}{2}x}$

general solution

$$\boxed{y = c_1 e^{-\frac{1}{2}x} + c_2 \underline{\underline{x}}\, e^{-\frac{1}{2}x}}$$