5.2 General Solutions of Linear Diff. Eqs

last time: $y'' + p(x)y' + g(x)y = f(x)$

cannot contain y (else nonlinear)

the associated homogeneous eq is → right side zero

$$y'' + p(x)y' + g(x)y = 0$$

solution to the homogeneous part:

$$y = c_1 y_1 + c_2 y_2$$

y_1 and y_2 are solutions so are their linear combos

if not homogeneous (right side not zero) then solution is

$$y = \underbrace{c_1 y_1 + c_2 y_2}_{\text{homogeneous part}} + \underbrace{y_p}_{\text{part due to right side}}$$

we add them:
The principle of superposition

we will also try to understand the <u>structure</u> of higher-order linear eqs.

for example, 3rd-order: $y''' = 0$

solution is easy: $y = \underbrace{c_1 x^2 + c_2 x + c_3}$

linear combo of
x^2, x, and 1
3 functions spanning the space

general solution of a linear 3rd-order

$y = c_1 y_1 + c_2 y_2 + c_3 y_3$

that is true for n^{th}-order $y^{(n)} + p_1(x) y^{(n-1)} + p_2(x) y^{(n-2)} + \dots + p_n(x) y = f(x)$

solution: $y = \underbrace{c_1 y_1 + c_2 y_2 + \dots + c_n y_n}_{\text{homogeneous part}} + \underbrace{y_p}_{\text{due to right side}}$

the n solutions are linearly independent

how to check? for example, for 3rd-order : y_1, y_2, y_3

linearly indp $\longrightarrow$ $c_1 y_1 + c_2 y_2 + c_3 y_3 = 0 \longleftrightarrow c_1 = c_2 = c_3 = 0$ for ALL x on an interval

deriv: $c_1 y_1' + c_2 y_2' + c_3 y_3' = 0$

deriv: $c_1 y_1'' + c_2 y_2'' + c_3 y_3'' = 0$

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

unique solution of $c_1 = c_2 = c_3 = 0$ if and only if

$$\underbrace{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}_{\text{Wronskian of } y_1, y_2, y_3} \neq 0$$

for example, are $y_1 = 1$, $y_2 = \sin^2 x$ and $y_3 = \cos^2 x$

linearly indp?

(this is a case we can tell by inspection:

$\sin^2 x + \cos^2 x = 1$

so, $c_1 \cdot 1 + c_2 \sin^2 x + c_3 \cos^2 x = 0$

has a solution w/ c_1, c_2, c_3 not all zero)

so they are not linearly indp

the Wronskian can give us the answer whether we can tell by inspection

$$W(1, \cos^2 x, \sin^2 x) = \begin{vmatrix} 1 & \cos^2 x & \sin^2 x \\ 0 & -2\cos x \sin x & 2\sin x \cos x \\ 0 & 2\sin^2 x - 2\cos^2 x & -(2\sin^2 x - 2\cos^2 x) \end{vmatrix}$$

expand along column 1

$$= \begin{vmatrix} -2\cos x \sin x & 2\sin x \cos x \\ 2\sin^2 x - 2\cos^2 x & -(2\sin^2 x - 2\cos^2 x) \end{vmatrix}$$

$= \cdots = 0$ Wronskian is <u>identically zero</u> for ALL x

So, these three functions are linearly dependent on all intervals of x

if $W \neq 0$ all the time, then the functions are linearly independent on at least some intervals of x

for 4 functions:

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

for n functions:

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ \vdots & & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & \cdots & y_n^{(n-1)} \end{vmatrix}$$

square matrix

Reduction of Order

sometimes a high-order eq. can be reduced to a lower-order one if we know at least one solution (somehow)

let's see a 2nd-order example

$$x^2 y'' + x y' - 9y = 0$$

linear but NOT constant coefficient
(this is an Euler's equation)

suppose that, somehow, we know one solution

$$y_1 = x^3$$

since this is 2nd-order, we know there is one more $y_2 = ?$

reduction of order method: $\boxed{y_2 = v(x)\, y_1}$

to find $v(x)$, sub $y_2 = v(x) y_1$ into the original diff. eq.

$$y_2 = v y_1 = v x^3$$

$$y_2' = 3vx^2 + v'x^3$$

$$y_2'' = 6vx + 6v'x^2 + v''x^3$$

sub into $x^2y'' + xy' - 9y = 0$

$$x^2(6vx + 6v'x^2 + v''x^3) + x(3vx^2 + v'x^3) - 9(vx^3) = 0$$

⋮

simplify

⋮

$$x^5v'' + 7v'x^4 = 0$$

$\rightarrow \quad v'' = -7v'x^{-1}$ $\qquad v'' = \dfrac{d(v')}{dx}$

$$\frac{d(v')}{dx} = -\frac{7}{x}(v')$$

separable in v' and x

$$\frac{1}{v'}\, d(v') = -\frac{7}{x}\, dx$$

⋮

solve separable

⋮

$$v' = c\, x^{-7}$$

constant

integrate: $V = \int c\, x^{-7} dx = -\frac{c}{6} x^{-6} + D$

$$V = -\frac{c}{6} x^{-6} + D \qquad y_2 = v y_1 = v x^3$$

choose ANY convenient c, D except those that lead to $V = 0$ or $V =$ constant

here, we choose $D = 0$, $c = -6$

$$V = x^{-6}$$

so, $y_2 = v y_1 = v x^3 = x^{-6} x^3 = \boxed{x^{-3}}$