5.2 General Solutions of Linear Diff. Eqs

2nd-order: $y'' + p(x)y' + g(x)y = f(x)$

do not contain y

the associated homogeneous eq. is

$$y'' + p(x)y' + g(x)y = 0$$

and it has solution $y = c_1 y_1 + c_2 y_2$

where y_1 and y_2 are two linearly indp. solutions

the general solution (including right side)

particular solution due to the right side ($f(x)$)

$$y = \underbrace{c_1 y_1 + c_2 y_2}_{\text{solution to the associated homogeneous eq.}} + y_p$$

this way of summing solutions is called the

Principle of Superposition

3rd-order: structurally identical to 2nd-order

for example, $y''' = 0$

solution is any 2nd-order polynomial

$$\underbrace{y = c_1 x^2 + c_2 x + c_3}_{\text{linear combo of } x^2,\ x,\ 1}$$

solution space spanned by 3 linearly indp functions

for $y''' + p_1(x) y'' + p_2(x) y' + p_3(x) y = f(x)$

solution structure is the same as in 2nd order

$$y = \underbrace{c_1 y_1 + c_2 y_2 + c_3 y_3}_{\text{homogeneous part}} + \underbrace{y_p}_{\text{due to } f(x)}$$

the structure continues for n^{th}-order

$$y^{(n)} + p_1(x) y^{(n-1)} + p_2(x) y^{(n-2)} + \cdots + p_n(x) y = f(x)$$

solution is $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_p$

linear indp: for two functions y_1 and y_2

are <u>dependent</u> on some interval of x

if $\underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}_{\text{Wronskian of } y_1 \text{ and } y_2} = 0$ for every x in that interval

if Wronskian is NOT zero all the time, then y_1 and y_2 are indp on at least some interval

for 3 functions, y_1, y_2, y_3, same idea

$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0 \rightarrow c_1 = c_2 = c_3 = 0$ only then they are <u>indp</u>

deriv: $c_1 y_1' + c_2 y_2' + c_3 y_3' = 0$

deriv: $c_1 y_1'' + c_2 y_2'' + c_3 y_3'' = 0$

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has unique solution of $c_1 = c_2 = c_3 = 0$ if and only if

$$\underbrace{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}_{\text{Wronskian of } y_1,\, y_2,\, y_3} \neq 0$$

for n functions, $y_1, y_2, y_3, \ldots, y_n$ the Wronskian follows the same pattern

$$W(y_1, y_2, \ldots, y_n) = \underbrace{\begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ & & \vdots & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & & y_n^{(n-1)} \end{vmatrix}}_{\text{matrix has to be square}}$$

Are $y_1 = 1$, $y_2 = \sin^2 x$ and $y_3 = \cos^2 x$ linearly indp?
(and on what interval(s))

$$W(1, \sin^2 x, \cos^2 x) = \begin{vmatrix} 1 & \sin^2 x & \cos^2 x \\ 0 & 2\sin x \cos x & -2\cos x \sin x \\ 0 & -(2\sin^2 x - 2\cos^2 x) & 2\sin^2 x - 2\cos^2 x \end{vmatrix}$$

expand along column 1

$$= 1 \cdot \begin{vmatrix} 2\sin x \cos x & -2\cos x \sin x \\ -(2\sin^2 x - 2\cos^2 x) & 2\sin^2 x - 2\cos^2 x \end{vmatrix} = \cdots = 0$$

zero on ALL intervals of x

these three functions are linearly dependent on ALL intervals of x

$\sin^2 x + \cos^2 x = 1$ for ALL x

$\sin^2 x + \cos^2 x - 1 = 0$ " "

$C_1 \sin^2 x + C_2 \cos^2 x + C_3 \cdot 1 = 0 \rightarrow \underbrace{C_1 = 1,\ C_2 = 1,\ C_3 = -1}_{\text{NOT all zeros} \rightarrow \text{dependent}}$

2nd-order homogeneous: $y'' + p(x)\,y' + g(x)\,y = 0$

if coeffs are constants: $y'' + ay' + by = 0$

then solutions are $\boxed{e^{r_1 x} \text{ and } e^{r_2 x}}$ where only true if coeffs are constant

r_1 and r_2 are roots of $\underbrace{r^2 + ar + b = 0}_{\text{characteristic eq.}}$

If p, g are not constants, solutions are harder to find.

if we somehow know one solution, y_1 then we can find the 2nd one by assuming $\boxed{y_2 = v(x)\, y_1}$

find $v \to$ find y_2 Reduction of Order

example $x^2 y'' + x y' - 9y = 0$

Suppose we know (somehow) that $y_1 = x^3$

find the second solution y_2

using Reduction of Order, we assume

$$y_2 = v(x)\, y_1 = v x^3$$

sub y_2 into the diff. eq.

$$y_2' = 3vx^2 + v'x^3$$

$$y_2'' = 6vx + 6v'x^2 + v''x^3$$

sub into $x^2y'' + xy' - 9y = 0$

$$x^2(6vx+6v'x^2+v''x^3)+x(3vx^2+v'x^3)-9vx^3=0$$

$$\cancel{6v}x^3+6v'x^4+v''x^5+\cancel{3v}x^3+v'x^4-\cancel{9v}x^3=0$$

$$x^5v''+7v'x^4=0 \qquad \text{find } v$$

$$v''=-7v'x^{-1} \qquad \text{let } u=v'$$

$$\frac{du}{dx}=-\frac{7u}{x} \qquad \text{1st-order separable in } u \text{ and } x$$

$$\frac{1}{u}\,du=-\frac{7}{x}\,dx$$

$$\ln u=-7\ln x+C$$

$$u=e^{-7\ln x+C}=e^{\ln x^{-7}}\cdot e^{C}=Cx^{-7}$$

$$v'=Cx^{-7}$$

integrate: $v=-\frac{C}{6}x^{-6}+D$

C, D are arbitrary : choose ANY convenient values
as long as Rc and $C \neq 0$

here, choose $D = 0$ and $C = -6 \rightarrow V = x^{-6}$

$y_2 = V y_1 = x^{-6} x^3 = x^{-3}$

$x^2 y'' + x y' - 9y = 0$ has a general solution

$$\boxed{y = c_1 x^3 + c_2 x^{-3}}$$