5.3 Homogeneous Eqs. with Constant Coefficients

homogeneous: right side zero

$$y''+5y'-2y=0$$
$$y'''+10y''-5y'+17=0$$
$$y^{(10)}-100y=0$$

find a y that is a constant multiple of its derivatives

$\rightarrow \boxed{y=e^{rx}}$ $\quad \underbrace{y'=re^{rx}}_{\text{r times itself}}$

if we substitute $y=e^{rx}$ (because it solves those eqs.) into them, we get ~~the~~ <u>characteristic eq.</u> that we solve to find r.

for example, $2y''-3y'=0$

$$y=e^{rx} \quad y'=re^{rx} \quad y''=r^2e^{rx}$$

$2r^2e^{rx} - 3re^{rx} = 0$

$e^{rx}(2r^2 - 3r) = 0 \qquad e^{rx} \neq 0$

so, $\boxed{2r^2 - 3r = 0}$ Characteristic eq.

we can get this directly by using coefficients of the differential eq. $\rightarrow$ coeff of y'' becomes coeff of r^2

" " y' " " " r

" " y " " " 1

$2r^2 - 3r = 0$

$r(2r-3) = 0 \rightarrow r = 0, \quad r = 3/2$ distinct roots

solutions: $y_1 = e^{0x} = 1$

$y_2 = e^{3/2 x}$

general solution: linear combo of them $\boxed{y = C_1 \cdot 1 + C_2 e^{3/2 x}}$

example $4y'' - 12y' + 9y = 0$

coeff of r^2 coeff of r coeff of 1

$$4r^2 - 12r + 9 = 0$$

$$(2r-3)(2r-3) = 0$$

$r = 3/2,\ 3/2$ repeated

Solutions: $y_1 = e^{3/2 x}$

$y_2 = x e^{3/2 x}$ multiply by x each time it is repeated

gen. solution: $\boxed{y = c_1 e^{3/2 x} + c_2 x e^{3/2 x}}$

$$y''' = 0$$

$$r^3 = 0 \longrightarrow r = 0,\ 0,\ 0$$

$$y_1 = e^{0x} = 1$$

$$y_2 = x e^{0x} = x$$

$$y_3 = x^2 e^{0x} = x^2$$

each repeat: one factor of x multiplied

$$y = c_1 + c_2 x + c_3 x^2$$

<u>example</u>

$$y^{(4)} - 8y''' + 16y'' = 0$$

$$r^4 - 8r^3 + 16r^2 = 0$$

$$r^2(r^2 - 8r + 16) = 0$$

$$(r)(r)(r-4)(r-4) = 0$$

$$r = 0,\ 0,\ 4,\ 4$$

$y_1 = e^{0x} = 1$
$y_2 = x e^{0x} = x$ } from $r = 0, 0$

$y_3 = e^{4x}$
$y_4 = x e^{4x}$ } from $r = 4, 4$

gen. solution: $\boxed{y = c_1 + c_2 x + c_3 e^{4x} + c_4 x e^{4x}}$

we can work backwards:

solution is $y = c_1 e^{-2x} + c_2 x e^{-2x} + c_3 x^2 e^{-2x}$

what is the differential eq?

$y_1 = e^{-2x} \longrightarrow r = -2$

$y_2 = \underline{x} e^{\underline{-2}x} \longrightarrow$ repeat of $r = -2$ (x in front $\rightarrow$ repeat)

$y_3 = x^2 e^{-2x} \longrightarrow r = -2$ (2nd repeat) characteristic eq: $(r+2)(r+2)(r+2)=0$

$$(r+2)(r^2+4r+4)=0$$

$$r^3+6r^2+12r+8=0$$

coeff of y''' coeff of y''

differential eq: $y'''+6y''+12y'+8y=0$

cubic characteristic eqs can be hard to solve
(no nice formula like quadratic formula)

<u>example</u> $y''' + 5y'' - 100y' - 500y = 0$

characteristic eq: $r^3 + 5r^2 - 100r - 500 = 0$

here, there is a pattern in coeffs: $\underbrace{1, 5}_{\text{mult. of } 5}, \underbrace{-100, -500}_{\text{mult. of } 5}$

$$r^3 + 5r^2 - 100r - 500 = 0$$

$$r^2(r+5) - 100(r+5) = 0$$

$$(r+5)(r^2 - 100) = 0$$

$$(r+5)(r+10)(r-10) = 0 \qquad r = -5, -10, 10$$

$$y = c_1 e^{-5x} + c_2 e^{-10x} + c_3 e^{10x}$$

example $3y''' + 4y'' - 5y' - 2y = 0$

$3r^3 + 4r^2 - 5r - 2 = 0$

coeff: 3, 4, -5, -2   no nice pattern like last time

turn into $(r-a)(\text{quadratic}) = 0$

one root

if we can find one root, then we just need to find that quadratic to proceed further

by inspection or trial and error

$3r^3 + 4r^2 - 5r \boxed{-2} = 0$

one root comes from this

$-2 = -2 \cdot 1 = -1 \cdot 2$

one root is -2 or 1 or -1 or 2

one of these is a root: $\pm 2, \pm 1$

pick one and try: $r = 1$

$3r^3 + 4r^2 - 5r - 2 = 0$ is $r=1$ a solution?

$3(1)^3 + 4(1)^2 - 5(1) - 2 = 0$ so $r=1$ <u>is</u> a root

$$\underbrace{(r-1)}_{\substack{r=1 \\ \text{from} \\ \text{trial and} \\ \text{error}}}\underbrace{(ar^2+br+c)}_{?} = 0$$

$$(r-1)(ar^2+br+c) = 3r^3+4r^2-5r-2$$

$$ar^2+br+c = \frac{3r^3-4r^2-5r-2}{r-1}$$

polynomial long division

that quadratic

$$\boxed{3r^2 + 7r + 2}$$

$$r - 1 \;\big)\; 3r^3 + 4r^2 - 5r - 2$$

$$-\left(3r^3 - 3r^2\right)$$

$$7r^2 - 5r - 2$$

$$-\left(7r^2 - 7r\right)$$

$$2r - 2$$

$$-\left(2r - 2\right)$$

$$0$$

$$(r-1)(3r^2+7r+2)=0$$

$$(r-1)(3r+1)(r+2)=0 \qquad r = 1,\ -1/3,\ -2$$

$$y = c_1 e^{x} + c_2 e^{-1/3 x} + c_3 e^{-2x}$$

next time: complex roots $\rightarrow$ $e^{ix} = ?$