# 5.3 Homogeneous Eqs. w/ Constant Coefficients (part 2)

$y'' + ay' + by = 0$ $\quad$ $a, b$ constants

solutions: $y = e^{rx}$ $\quad$ $r$: solutions of characteristic eq.

$$r^2 + ar + b = 0$$

two cases: $r$'s are real and distinct $\rightarrow r_1, r_2 \quad r_1 \neq r_2$

$$y_1 = e^{r_1 x} \qquad y_2 = e^{r_2 x}$$

$r$'s are repeated $\quad r_1 = r_2 = r$

$$y_1 = e^{rx} \qquad y_2 = x e^{rx}$$

today: $r$'s are complex

start w/ : $y'' + y = 0$

characteristic eq $r^2 + 1 = 0$ $\qquad r^2 = -1$

$r = \pm\sqrt{-1} = \pm i$

$i^2 = -1$

solutions: $e^{rx}$

$\hookrightarrow e^{ix}, \; e^{-ix}$ $\qquad e^{ix} = ?$

$y'' + y = 0 \rightarrow y'' = -y$

solution is a function that is the negative of its 2nd derivative

exponential? $y = e^{ax}$ $\quad y' = ae^{ax}$ $\quad y'' = a^2 e^{ax}$

same sign!

So, NOT exponential

but $y = \cos x$, $y' = -\sin x$, $y'' = -\cos x$

$y = \sin x$, $y' = \cos x$, $y'' = -\sin x$

so, $e^{ix}$ and $e^{-ix}$ must be related to $\cos x$ and $\sin x$

how?

$$e^{ix} = ?$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$ (Taylor series of $e^x$)

now replace $x$ with $ix$

| | |
|---|---|
| $i = i$ | $i^5 = i$ |
| $i^2 = -1$ | $i^6 = -1$ |
| $i^3 = -i$ | $i^7 = -i$ |
| $i^4 = 1$ | $i^8 = 1$ |

cycle repeats

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

$$\vdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots$$

$$= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots\right)}_{\cos x} + i\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots\right)}_{\sin x}$$

$$\boxed{e^{ix} = \cos x + i \sin x}$$ Euler's Formula

$$e^{-ix} = \cos x - i \sin x$$

example $y'' + 100y = 0$

$r^2 + 100 = 0 \qquad r = \pm\sqrt{-100} = \pm i\sqrt{100} = \pm 10i$

solutions: $e^{10ix} = e^{i(10x)} = \cos(10x) + i\sin(10x)$ — ①

$e^{-10ix} = e^{i(-10x)} = \cos(-10x) + i\sin(-10x)$

$= \cos(10x) - i\sin(10x)$ — ②

these solutions are complex but we know solutions should be real-valued

① + ② divided by 2 → $\cos(10x)$

① − ② divided by $2i$ → $\sin(10x)$

} simply the real and imaginary parts of $e^{rx}$

these are taken to be the two solutions (real-valued) that span the solution space

so, $\boxed{y_1 = \cos(10x) \qquad y_2 = \sin(10x)}$

general solution: $\boxed{y = C_1\cos(10x) + C_2\sin(10x)}$

so, the solutions are just the real and imaginary parts of $e^{rx}$ w/ $r$ being complex.

<u>example</u> $y'' - 6y' + 25y = 0$

$r^2 - 6r + 25 = 0$ cannot factor this

$$r = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(25)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{-64}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$$

form one solution, grab the real and imaginary parts

use $r = 3 + i4$

$$e^{rx} = e^{(3+4i)x} = e^{3x} e^{4ix} = e^{3x} e^{i(4x)}$$

$$e^{ix} = \cos x + i \sin x$$

$$e^{3x} e^{i(4x)} = e^{3x} ( \cos 4x + i \sin 4x )$$

$$= \underbrace{\left(e^{3x} \cos 4x\right)}_{\text{real part}} + i \underbrace{\left(e^{3x} \sin 4x\right)}_{\text{imaginary part}}$$

use the real and imag parts to span solution space

$y_1 = e^{3x} \cos 4x \qquad y_2 = e^{3x} \sin 4x$

general solution: $\boxed{y = c_1 e^{3x} \cos 4x + c_2 e^{3x} \sin 4x}$

<u>example</u> $6y^{(4)} + 11y'' + 4y = 0$

$$6r^4 + 11r^2 + 4 = 0$$

$$6(r^2)^2 + 11(r^2) + 4 = 0$$

$$6u^2 + 11u + 4 = 0 \qquad u = r^2$$

$$u = \frac{-11 \pm \sqrt{11^2 - 4(6)(4)}}{2(6)} = \frac{-11 \pm \sqrt{25}}{12}$$

$$= \frac{-11 \pm 5}{12} = -\frac{4}{3},\ -\frac{1}{2}$$

$u = r^2$

$$r^2 = -\frac{4}{3}, \quad r^2 = -\frac{1}{2}$$

$$r = \frac{2}{\sqrt{3}}i,\ -\frac{2}{\sqrt{3}}i,\ \frac{1}{\sqrt{2}}i,\ -\frac{1}{\sqrt{2}}i$$

form one solution using $r = \pm \frac{2}{\sqrt{3}} i$. grab real and imag parts

$$e^{\frac{2}{\sqrt{3}} i x} = \left[\cos\left(\frac{2}{\sqrt{3}} x\right)\right] + i\left[\sin\left(\frac{2}{\sqrt{3}} x\right)\right]$$

$y_1 = \cos\left(\frac{2}{\sqrt{3}} x\right)$

$y_2 = \sin\left(\frac{2}{\sqrt{3}} x\right)$

repeat w/ $r = \pm \frac{1}{\sqrt{2}} i$

$$e^{\frac{1}{\sqrt{2}} i x} = \left[\cos\left(\frac{1}{\sqrt{2}} x\right)\right] + i\left[\sin\left(\frac{1}{\sqrt{2}} x\right)\right]$$

$y_3 = \cos\left(\frac{1}{\sqrt{2}} x\right)$

$y_4 = \sin\left(\frac{1}{\sqrt{2}} x\right)$

general solution: $y = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$

$$\boxed{y = c_1 \cos\left(\frac{2}{\sqrt{3}} x\right) + c_2 \sin\left(\frac{2}{\sqrt{3}} x\right) + c_3 \cos\left(\frac{1}{\sqrt{2}} x\right) + c_4 \sin\left(\frac{1}{\sqrt{2}} x\right)}$$

example

$$y''' - 5y'' + 100y' - 500y = 0$$

$$r^3 - 5r^2 + 100r - 500 = 0$$

$$r^2(r-5) + 100(r-5) = 0$$

$$(r-5)(r^2+100) = 0$$

$$r_1 = 5, \quad \underbrace{r_2 = 10i, \quad r_3 = -10i}_{\text{complex always come in pairs}}$$

$$y = c_1 e^{5x} + c_2 \cos(10x) + c_3 \sin(10x)$$