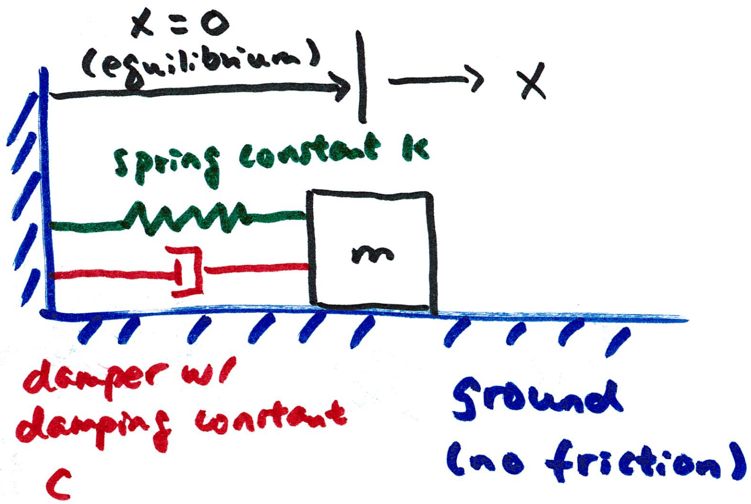
5.4 Mechanical Vibrations

mass-spring-damper



block mass m

spring w/ spring constant k

damper (dashpot) w/
damping constant c

$x=0$ is equilibrium

Give $x(0)$ and/or $x'(0)$, find $x(t)$

displacement from equilibrium

spring resists displacement : $F_s = -kx$ (Hooke's Law)

damper resists velocity : $F_d = -cx'$

Newton's 2nd Law: $\sum F = mx''$

$$-kx - cx' = mx'' \rightarrow \boxed{mx'' + cx' + kx = 0}$$

Linear constant coeff.

example mass 8 kg

no damper

spring such that a force of 40N stretches it from equilibrium by 5 cm.

$mx'' + cx' + kx = 0$ $m = 8$ $c = 0$

k to be found

Hooke's Law: $F_s = kx$ displacement from equilibrium

$40\,N = k \cdot (0.05\,m)$

meters

$k = 800$ N/m

$8x'' + 800x = 0$

$x'' + 100x = 0$ $r^2 + 100 = 0$ $r = 10i, -10i$

$x(t) = c_1 \cos(10t) + c_2 \sin(10t)$

initial conditions: $x(0)=0$, $x'(0)=10$

no initial displacement

initial velocity 10 m/s

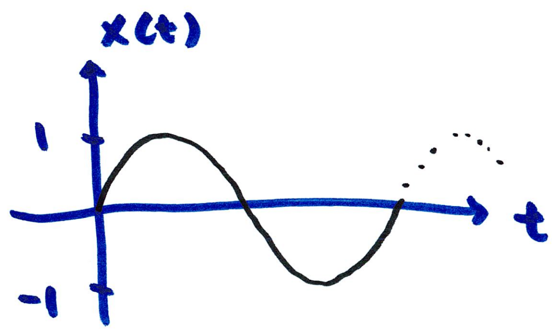
$$x(t) = C_1 \cos(10t) + C_2 \sin(10t)$$

$$x'(t) = -10C_1 \sin(10t) + 10C_2 \cos(10t)$$

$$x(0)=0 \rightarrow 0 = C_1$$

$$x'(0)=10 \rightarrow 10 = 10C_2 \rightarrow C_2 = 1$$

$$\boxed{x(t) = \sin(10t)}$$



period: $\frac{2\pi}{\text{freq}} = \frac{2\pi}{10} = \frac{\pi}{5}$ seconds

frequency (circular): 10 rad/s

amplitude: 1

frequency (linear): $\frac{1}{\text{period}} = \frac{5}{\pi}$ cycles per second (Hz) same setup, but $x(0)=2$ and $x'(0)=10$

$$x(t) = c_1 \cos(10t) + c_2 \sin(10t)$$

$$x'(t) = -10c_1 \sin(10t) + 10c_2 \cos(10t)$$

$$x(0)=2 \to \cdots \to c_1 = 2$$

$$x'(0)=10 \to \cdots \to c_2 = 1$$

now particular solution is $\boxed{x(t) = 2\cos(10t) + \sin(10t)}$

same freq: $\cancel{\frac{\pi}{5}}$ 10 rad/s

same period: $\frac{\pi}{5}$

amplitude ?

omega (freq)

express in alternate form: $\boxed{x(t) = A\cos(\omega t) + B\sin(\omega t)}$ $\longleftrightarrow$ $\boxed{x(t) = C\cos(\omega t - \delta)}$

amplitude

delta (phase shift)

using the identity $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

after some algebra, we get $\boxed{C = \sqrt{A^2+B^2}}$ $\boxed{\delta = \tan^{-1}\left(\frac{B}{A}\right)}$

here, $C = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\delta = \tan^{-1}\left(\frac{1}{2}\right) \approx 0.464$

$\boxed{x(t) = \sqrt{5}\cos(10t - 0.464)}$ amplitude $\sqrt{5}$

back to $mx'' + cx' + kx = 0$

$m, c, k \neq 0$ w/ damper

$mr^2 + cr + k = 0$

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

$c^2 - 4km$ is called the discriminant

it determines the type of solution

if $c^2 - 4km > 0$ ($c^2 > 4km$ or $c > \sqrt{4km}$)

roots of char. eq are <u>real and distinct</u>

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

strong damper $\longrightarrow$ <u>overdamped</u>

if $c^2 - 4km = 0$ ($c^2 = 4km$)

roots are <u>real and repeated</u>

$$x(t) = c_1 e^{rt} + c_2 t e^{rt}$$

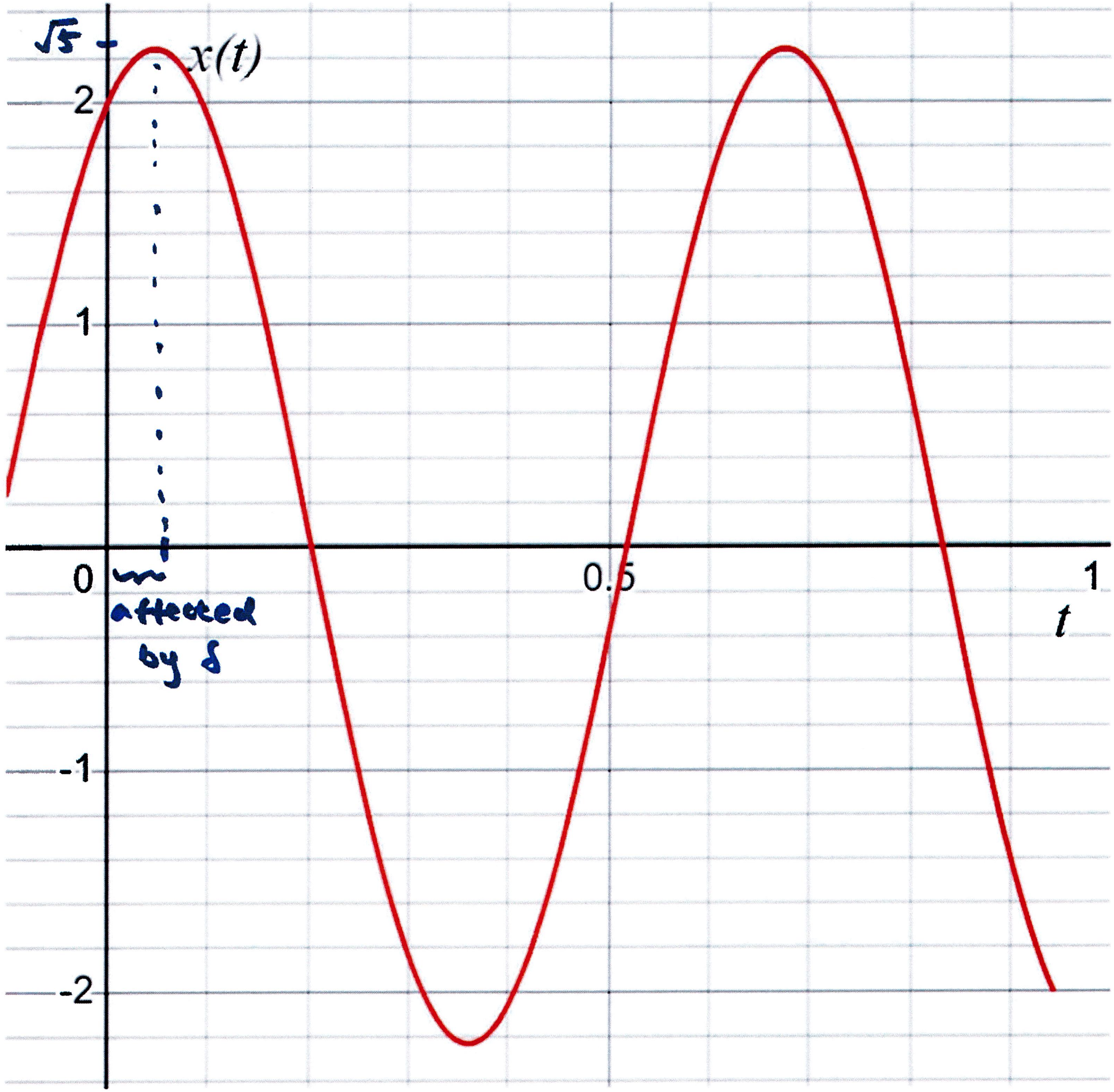
damped "just right" $\longrightarrow$ <u>critically damped</u>

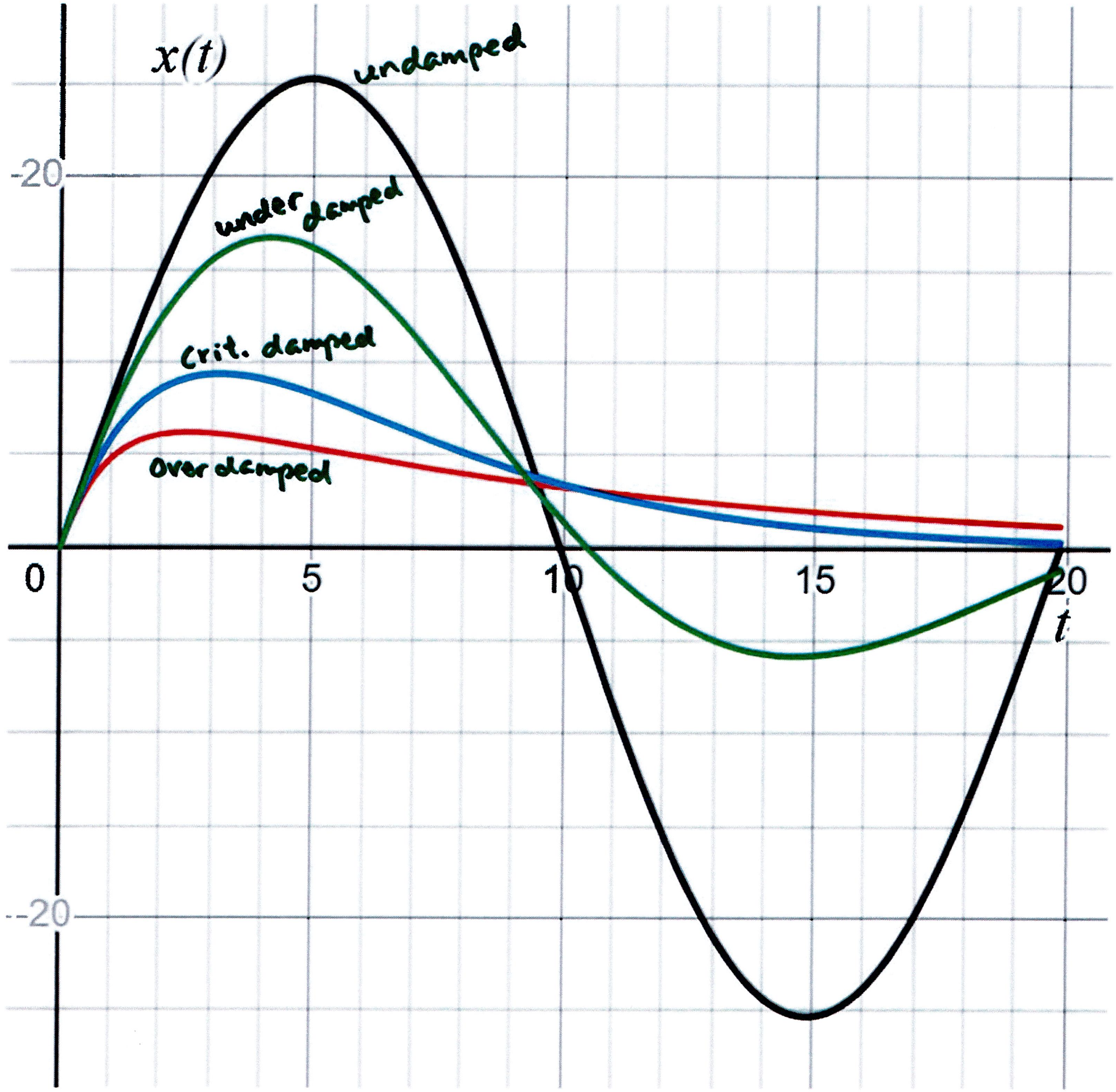
if $c^2 - 4km < 0$ ($c^2 < 4km$)

roots : $r = a \pm bi$

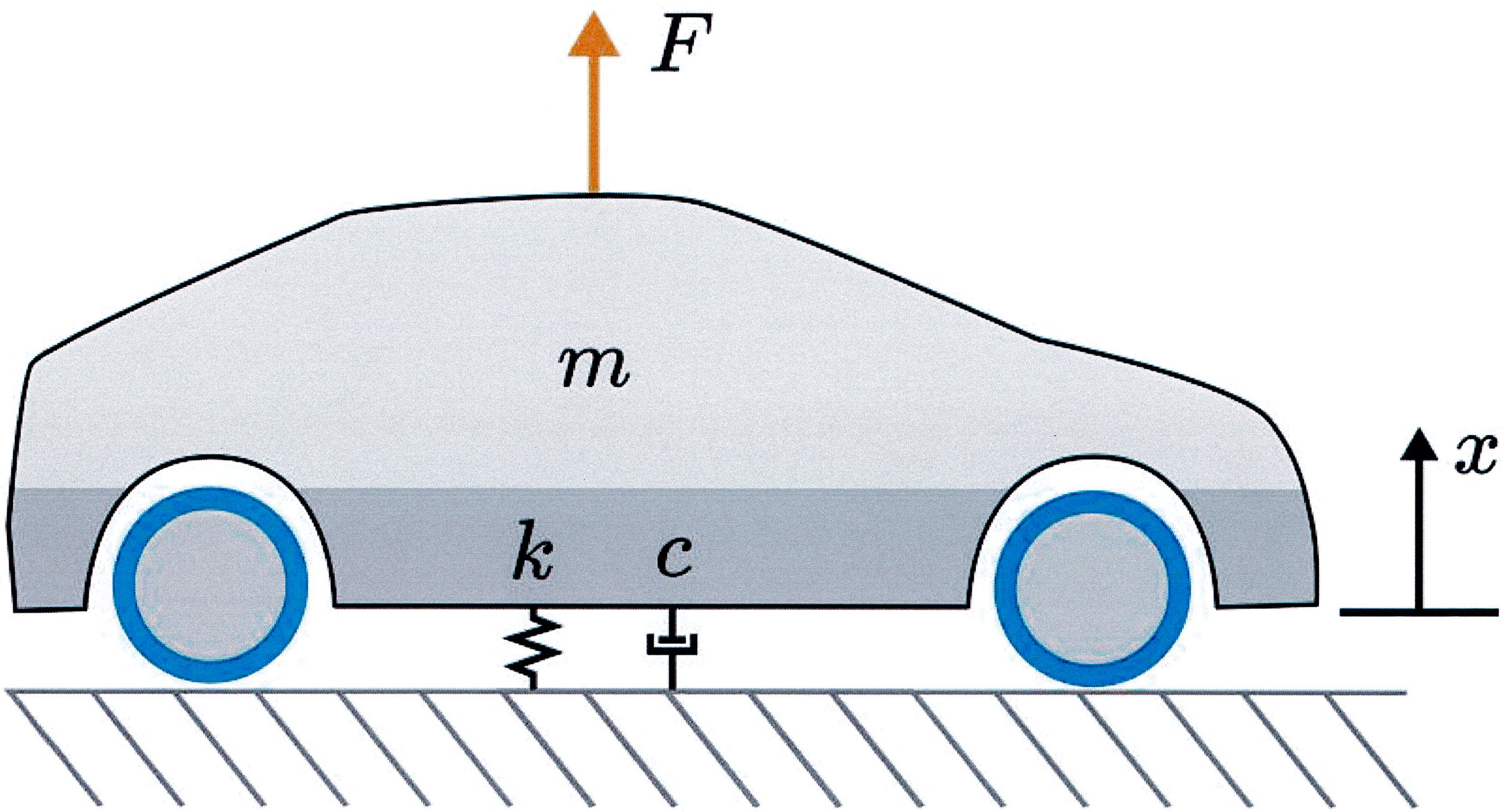
$$x(t) = c_1 e^{at} \cos(bt) + c_2 e^{at} \sin(bt)$$

weak damper $\rightarrow$ <u>underdamped</u> only case w/ oscillations


$\sqrt{5}$
$x(t)$
2
1
0
-1
-2
0.5
1
t
affected by δ

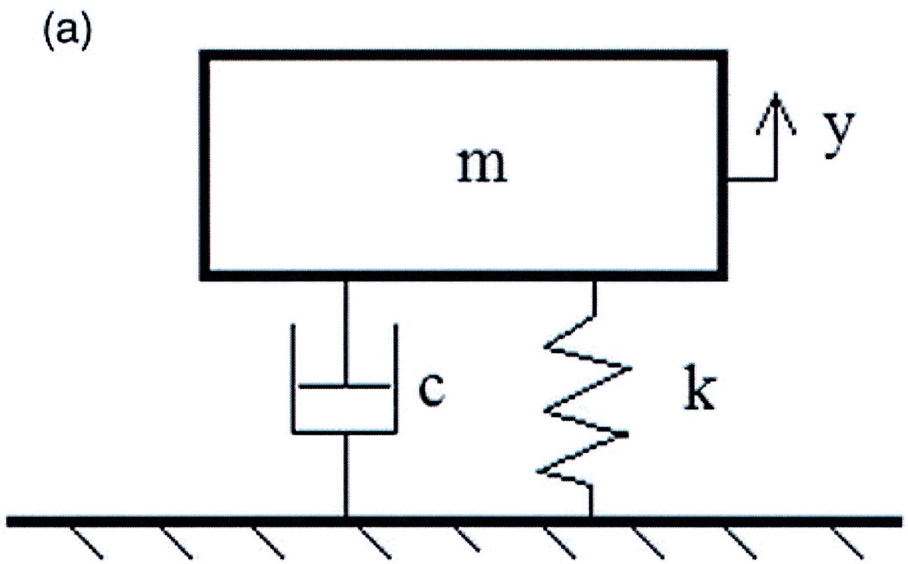

x(t)
undamped
-20
under damped
crit. damped
over damped
0
5
10
15
20
t
--20

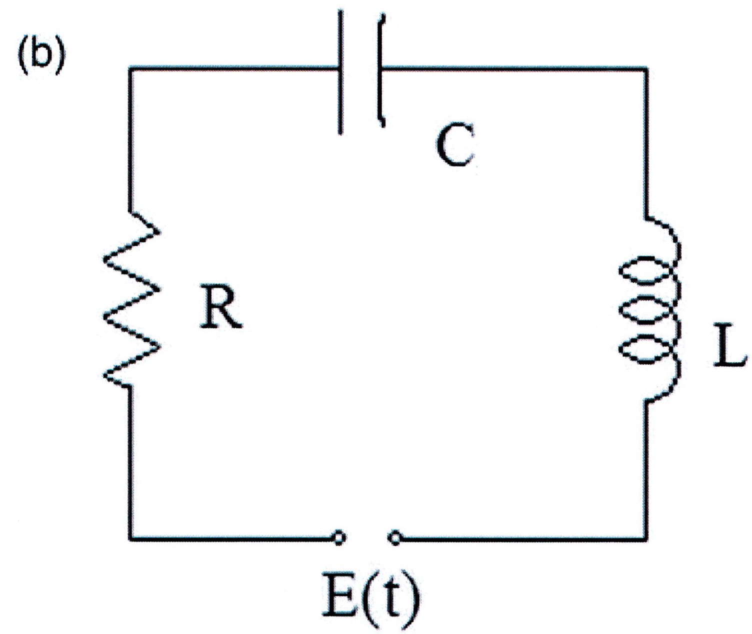
Suspension modeled as
mass-spring-damper



mass-spring-damper

RLC circuit





$$my'' + cy' + ky = F(t)$$

↑ external force

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

Q : charge