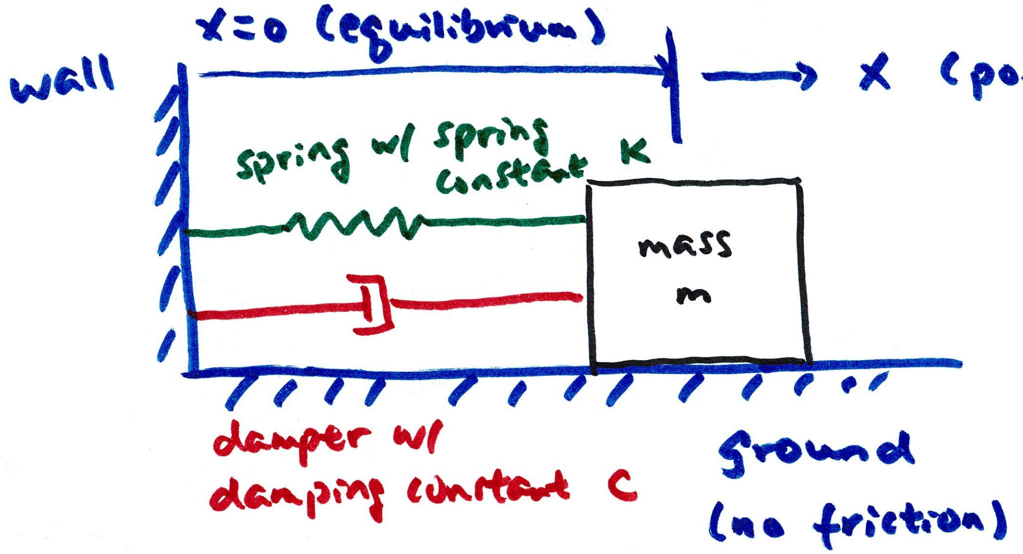
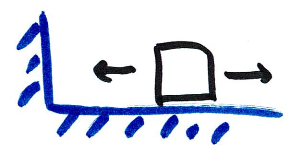
5.4 Mechanical Vibrations

mass-spring-damper system



if displaced or given a velocity from equilibrium, the block will have a certain motion



spring provides a force in the opposite direction of the displacement

Hooke's Law: $F_s = -kx$

damper resists velocity $F_d = -cx'$

equation of motion: Newton's 2nd Law

$$\sum F = m x''$$

$$-kx - cx' = mx'' \rightarrow \boxed{mx'' + cx' + kx = 0}$$

constant coeff linear eq.

<u>example</u> mass 8 kg, no damper

spring such that a force of 40 N stretches it by 5 cm.

Initial displacement $\rightarrow x(0) = 0$

initial velocity $\rightarrow x'(0) = 10$ m/s

$$m x'' + \cancel{c}^{\,0} x' + kx = 0 \qquad m = 8$$

k: Hooke's Law $\qquad F_s = kx$ (no negative here because we are dealing w/ magnitude)

$$40 = k \cdot (0.05)$$

$\leftarrow$ 5 cm in m

$$k = 800 \quad (N/m)$$

$$\rightarrow 8x'' + 800x = 0$$

$x'' + 100x = 0$

char. eq: $r^2 + 100 = 0 \quad r = 10i, \quad r = -10i$

$x(t) = C_1 \cos(10t) + C_2 \sin(10t)$ general

$x(0) = 0$

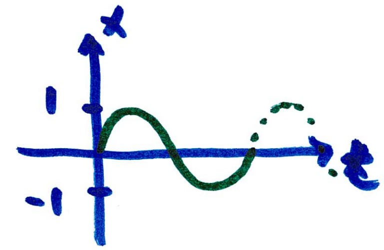
$x'(0) = 10$

$x'(t) = -10C_1 \sin(10t) + 10C_2 \cos(10t)$

$x(0) = 0 \rightarrow 0 = C_1 \cos(0) + C_2 \sin(0) = C_1$

$x'(0) = 10 \rightarrow 10 = 10C_2 \rightarrow C_2 = 1$

$\boxed{x(t) = \sin(10t)}$



amplitude: 1

circular frequency: 10 rad/s

period: $\dfrac{2\pi}{\text{circ. freq}} = \dfrac{2\pi}{10} = \dfrac{\pi}{5}$ seconds

frequency: $\dfrac{1}{\text{period}} = \dfrac{5}{\pi}$ Hz (cycles per sec) change initial conditions: $x(0) = 2$, $x'(0) = 10$

same gen. solution

$$x(t) = C_1 \cos(10t) + C_2 \sin(10t)$$
$$x'(t) = -10 C_1 \sin(10t) + 10 C_2 \cos(10t)$$

$x(0) = 2 \rightarrow 2 = C_1$

$x'(0) = 10 \rightarrow 10 = -20 \sin(0) + 10 C_2 \rightarrow C_2 = 1$

$$\boxed{x(t) = 2\cos(10t) + \sin(10t)}$$

same freq: $\frac{2\pi}{10} = \frac{\pi}{5}$ rad/s

same period: $\frac{5}{\pi}$ seconds

amplitude changed

to find amplitude, we rewrite ~~x(t)~~ a

$x(t) = A \cos(\omega t) + B \sin(\omega t)$ omega (circ. freq)

as

$x(t) = C \cos(\omega t - \delta)$ delta (phase shift)

using the identity $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

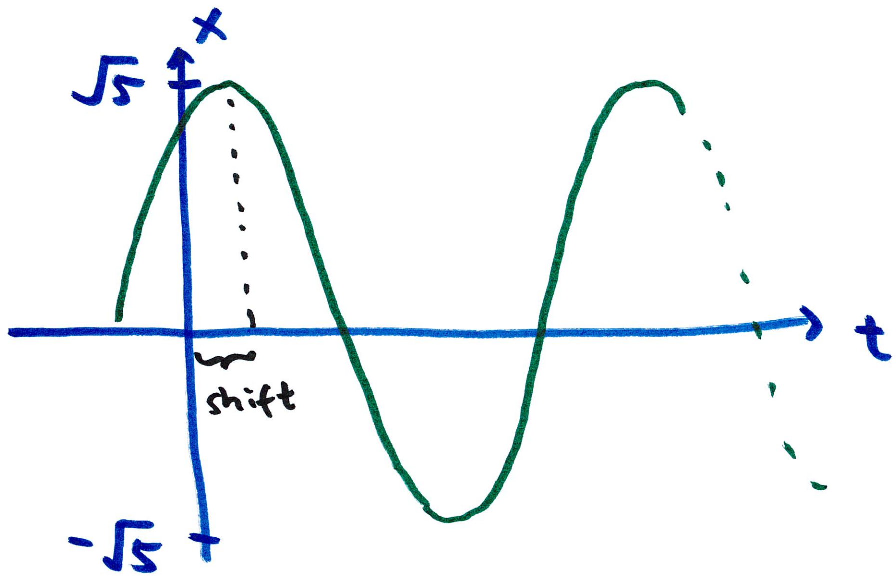
after a bit of algebra, we get

$$\boxed{C = \sqrt{A^2+B^2}} \qquad \boxed{\delta = \tan^{-1}\left(\frac{B}{A}\right)}$$

$x(t) = 2\cos(10t) + \sin(10t)$

$$\rightarrow \boxed{x(t) = \sqrt{5}\cos(10t - 0.464)}$$

amplitude $\sqrt{5}$



w/o damper, the oscillation lasts forever now let's add the damper

$$mx'' + cx' + kx = 0 \qquad m, c, k \neq 0 \ (\text{all} > 0)$$

$$mr^2 + cr + k = 0$$

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

$c^2 - 4km \rightarrow$ discriminant

it determines the type of solution

if $c^2 - 4km > 0 \rightarrow c^2 > 4km$

roots are real and distinct r_1, r_2

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

<u>overdamped</u> (strong damper) both roots are real so <u>no oscillation</u>

if $c^2 - 4km = 0 \rightarrow c^2 = 4km$

roots are real and repeated

$$x(t) = c_1 e^{rt} + c_2 t e^{rt}$$

<u>critically damped</u> (damped "just right" to eliminate oscillation)

if $c^2 - 4km < 0 \rightarrow c^2 < 4km$

roots are complex: $r = a + bi$

$$x(t) = c_1 e^{at} \cos(bt) + c_2 e^{at} \sin(bt)$$

<u>underdamped</u> (weak damper)

the <u>only</u> case w/ oscillations

<u>example</u> $mx'' + cx' + kx = 0$ $x(0) = 0,\ x'(0) = 8$

case 1: $c = 11,\ m = 10,\ k = 1$ c^2 vs $4km$

overdamped

$$x(t) = \frac{80}{9} e^{-\frac{1}{10}t} - \frac{80}{9} e^{-t}$$

as $t \to \infty$, $x \to 0$ (equilibrium) no oscillations

case 2: $c = \sqrt{40},\ m = 10,\ k = 1$

critically damped

$$x(t) = 8t\, e^{-t/\sqrt{10}}$$

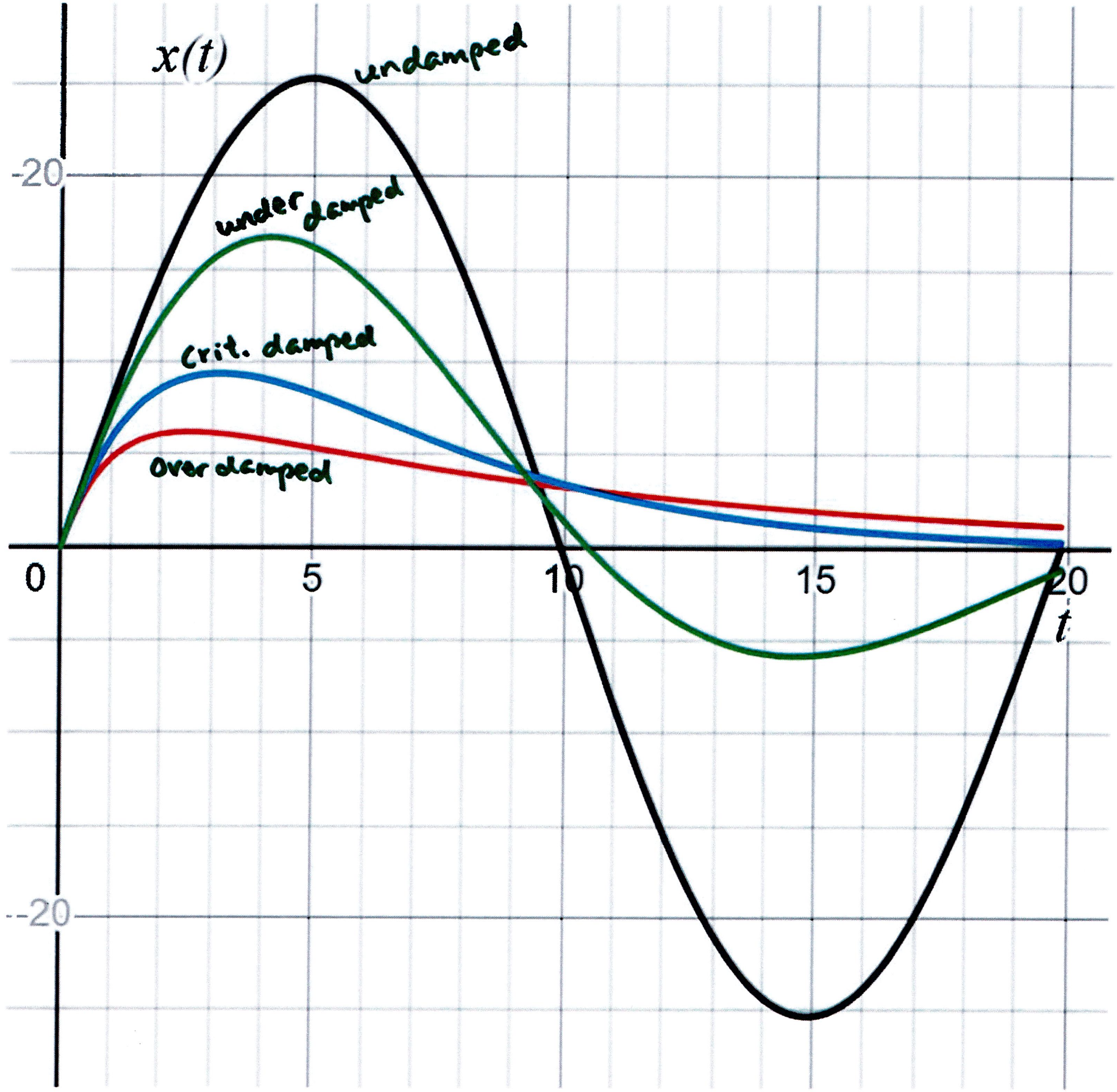
again, $t \to \infty$, $x \to 0$ no oscillations

case 3: $c = 2,\ m = 10,\ k = 1$

underdamped

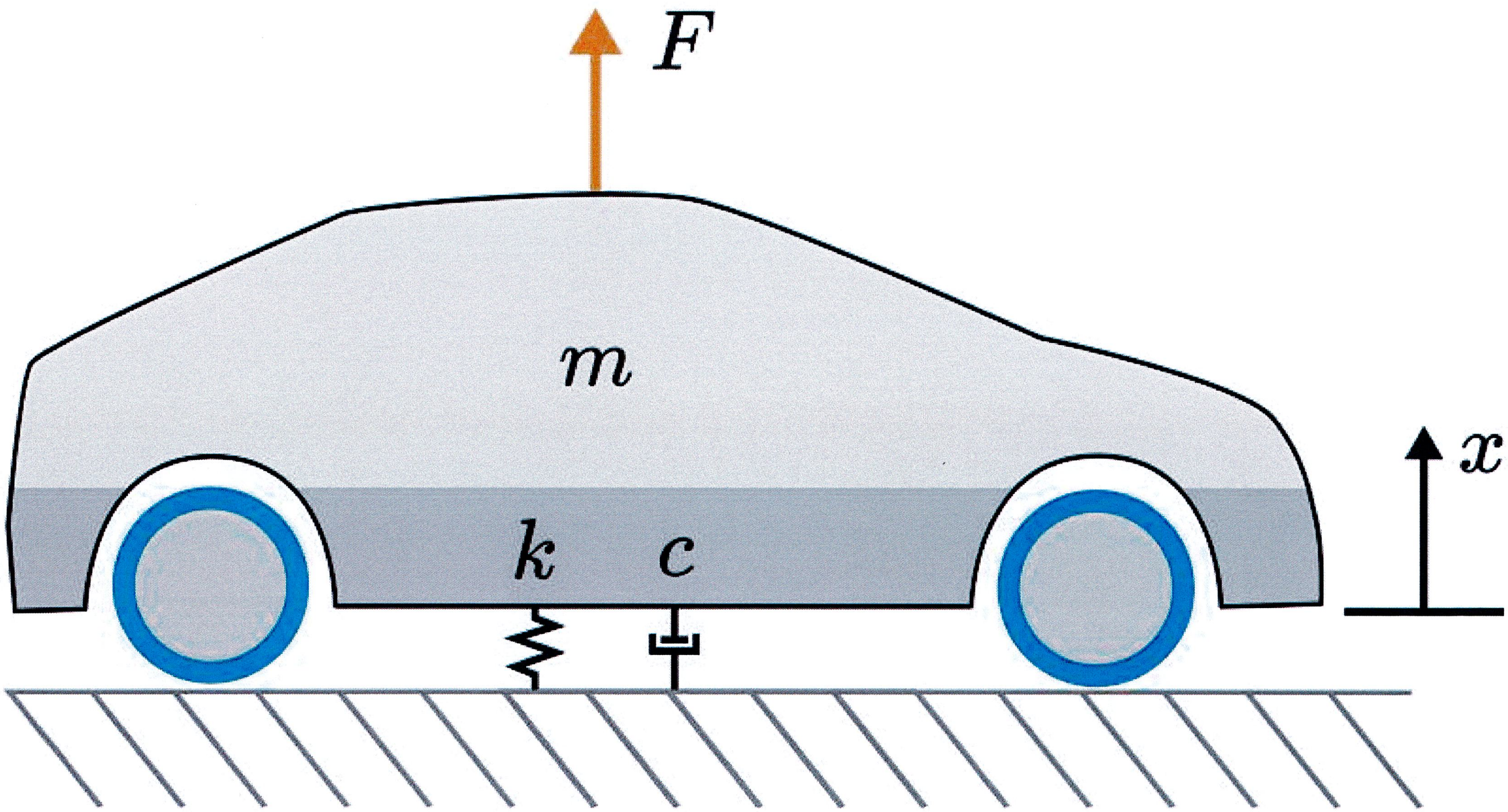
$$x(t) = \frac{80}{3} e^{-\frac{1}{10}t} \sin\left(\frac{3}{10} t\right)$$

$\lim_{t \to \infty} x = 0$ but w/ oscillations

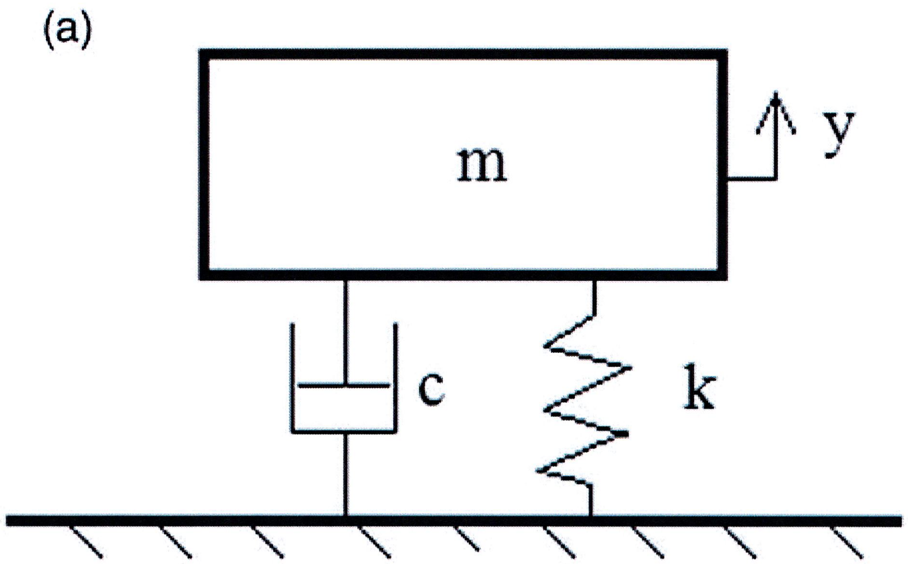


x(t)
undamped
20
under damped
crit. damped
over damped
0
5
10
15
20
t
-20

Suspension modeled as
mass-spring-damper



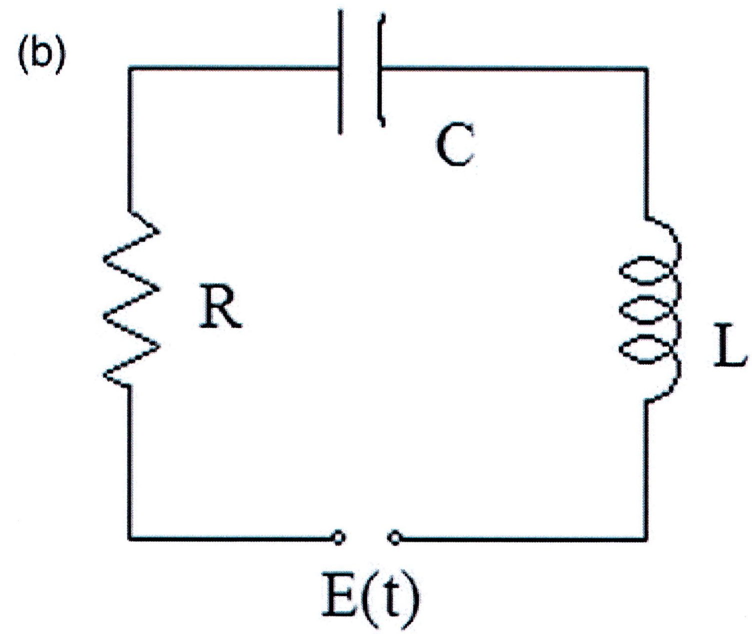
mass-spring-damper



$my'' + cy' + ky = F(t)$

external force

RLC circuit



$LQ'' + RQ' + \frac{1}{C}Q = E(t)$

Q : charge