5.5 Nonhomogeneous Eqs: Undetermined Coefficients

$$y'' + a\underbrace{\,}_{}y' + by = f(x)$$

constants

if 0 → homogeneous (solution in 5.3)

if ≠0 → nonhomogeneous

because eq. is linear → superposition applies

solution is $y = y_c + y_p$

particular solution (contribution from nonzero $f(x)$)

complementary solution (the homogeneous part, solve pretending $f(x) = 0$)

The method of undetermined coeffs is one method to find y_p

effective if $f(x)$ is polynomial

exponential

sine and or cosine

hyperbolic sine and hyp cosine

basic idea: "guess" the form of y_p based on the form of $f(x)$

<u>example</u> $y'' - 4y = 3x$

solution: $y = y_c + y_p$

y_c : solution to $y'' - 4y = 0$

$r^2 - 4 = 0 \quad r = \pm 2$

$y_c = c_1 e^{2x} + c_2 e^{-2x}$

y_p : "guess" a function w/ <u>unknown coeff</u> that has the <u>same form</u> as $f(x)$

here, $f(x) = 3x \longrightarrow$ first-deg polynomial

so, we "guess" $y_p = Ax + B$ 1st-deg polynomial w/ unknown A, B

y_p is itself a solution, so we sub it into the eq. to find the undetermined coeffs.

$$y'' - 4y = 3x \qquad \left\{ \begin{array}{l} y_p = Ax + B \\ y_p' = A \\ y_p'' = 0 \end{array} \right.$$

$$0 - 4(Ax + B) = 3x$$

$$\underline{-4A}x \; \underline{-4B} = \underline{3}x + \underline{0}$$

$$-4A = 3$$

$$-4B = 0$$

so, $A = -\frac{3}{4}$, $B = 0$

$$\boxed{y_p = -\frac{3}{4}x}$$

general solution: $y = y_c + y_p = \boxed{c_1 e^{2x} + c_2 e^{-2x} + \left(-\frac{3}{4}x\right)}$

example

$$y'' - y' - 2y = 3e^x$$

$y = y_c + y_p$ $\qquad$ y_c: solution to $y'' - y' - 2y = 0$

$$\vdots$$

$$y_c = c_1 e^{-x} + c_2 e^{2x}$$

y_p: matches the form of $f(x) = 3e^x$

$$\left.\begin{aligned} y_p &= Ae^x \\ y_p' &= Ae^x \\ y_p'' &= Ae^x \end{aligned}\right\} \text{ sub into } y'' - y' - 2y = 3e^x$$

$$Ae^x - Ae^x - 2Ae^x = 3e^x$$

$$-2Ae^x = 3e^x \qquad A = -\frac{3}{2}$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{2} e^x}$$

deriv of polynomial $\rightarrow$ polynomial

deriv of exponential $\rightarrow$ exponential

we need the derivs of y_p to keep the same form

for this method to work

if $f(x) = \cos x$ and we guess $y_p = A \cos x$

$y_p' = -A \sin x$ NOT cosine any more!

same if $f(x) = \sin x$

however, if we always guess $y_p = A \cos x + B \sin x$

then $y_p' = -A \sin x + B \cos x$

remains combo of cosine and sine (great!)

so, even if right side only has cosine or sine, we

ALWAYS include both in y_p.

<u>example</u> $y'' - y' - 2y = \cos x$

left side same as last example

so, $y_c = c_1 e^{-x} + c_2 e^{2x}$

y_p needs <u>BOTH</u> cosine and sine

$$\left.\begin{aligned} y_p &= A\cos x + B\sin x \\ y_p' &= -A\sin x + B\cos x \\ y_p'' &= -A\cos x - B\sin x \end{aligned}\right\} \text{ sub into } y'' - y' - 2y = \cos x$$

$$-A\cos x - B\sin x + A\sin x - B\cos x - 2A\cos x - 2B\sin x = \cos x$$

$$(-3A - B)\cos x + (-3B + A)\sin x = 1\cdot\cos x + 0\cdot\sin x$$

$$\left.\begin{aligned} -3A - B &= 1 \\ -3B + A &= 0 \end{aligned}\right\} \cdots \quad A = -\frac{3}{10} \quad B = -\frac{1}{10}$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{10}\cos x - \frac{1}{10}\sin x}$$

example $\quad y'' - y' - 2y = xe^{3x}$

$y = y_c + y_p \qquad y_c = c_1 e^{-x} + c_2 e^{2x}$

$$f(x) = xe^{3x}$$

$$= xe^{3x} + 0 \cdot e^{3x} = (\underbrace{x + 0}_{\text{1st-order polynomial}})e^{3x}$$

exponential

y_p matches form: $\quad y_p = (\overbrace{Ax + B})e^{3x}$

$y_p' = \cdots$

$y_p'' = \cdots$

sub into $y'' - y' - 2y = xe^{3x}$

$\vdots$

$y_p = \left(\frac{1}{4}x - \frac{5}{16}\right)e^{3x}$

$$\boxed{y = c_1 e^{-x} + c_2 e^{2x} + \left(\tfrac{1}{4}x - \tfrac{5}{16}\right)e^{3x}}$$

problem: f(x) matches the form of at least one of the functions in y_c

<u>example</u> $y'' + 100y = \cos(10x)$

y_c : $r^2 + 100 = 0 \quad r = \pm 10i$

$y_c = c_1 \cos(10x) + c_2 \sin(10x)$

$f(x) = \cos(10x)$

guess $y_p = A\cos(10x) + B\sin(10x)$

Same! Not indp.

fix: same as w/ repeated roots: toss x at it until problem goes away

correct $y_p = Ax\cos(10x) + Bx\sin(10x)$

then sub into eq. as usual