5.5 Nonhomogeneous Eqs: Undetermined Coefficients

$y'' + a y' + b y = f(x)$

a, b: constants

nonhomogeneous if $f(x) \neq 0$
homogeneous if $f(x) = 0$

the eq is linear so principle of superposition applies

$\rightarrow \quad y = y_c + y_p$

y_p: particular solution (from $f(x)$)

y_c: complementary solution (solution of $y'' + ay' + by = 0$)

method of undetermined coeffs: form of y_p matches the form of $f(x)$ but with different coefficients.

good for: polynomial $f(x)$
exponential
sine and cosine

example $y'' - 4y = 3x$

Solution is $y = y_c + y_p$

y_c : solution of $y'' - 4y = 0$

$r^2 - 4 = 0 \quad r = -2, 2$

$y_c = c_1 e^{-2x} + c_2 e^{2x}$

y_p : assume a form for y_p that is the same as $f(x)$

here, $f(x) = 3x = \underbrace{3x + 0}_{\text{first-deg polynomial}}$

so, we assume $y_p = Ax + B$

undetermined coeffs

$$y'' - 4y = 3x$$

sub into

$y_p = Ax + B$ is a solution so it satisfies the diff. eq.

$$y_p' = A$$

$$y_p'' = 0$$

$$0 - 4(Ax+B) = 3x$$

$$\underline{-4Ax} \; \underline{-4B} = \underline{3}x + \underline{0}$$

same

same

$-4A = 3$ so $A = -\frac{3}{4}$

$-4B = 0$ so $B = 0$

so, $y_p = -\frac{3}{4}x$

general solution:

$$\boxed{y = \underbrace{c_1 e^{-2x} + c_2 e^{2x}}_{\text{if right side is 0}} + \underbrace{\left(-\tfrac{3}{4}\right)x}_{\text{from right side } f(x) = 3x}}$$

example

$$y'' - y' - 2y = 3e^x$$

$y = y_c + y_p \qquad y_c$: solve $y'' - y' - 2y = 0$

$\vdots$

$$y_c = c_1 e^{-x} + c_2 e^{2x}$$

y_p : form of $f(x) = 3e^x$ is $y_p = Ae^x$

sub $y_p = Ae^x \qquad y_p' = Ae^x \qquad y_p'' = Ae^x$

into $y'' - y' - 2y = 3e^x$

$$Ae^x - Ae^x - 2Ae^x = 3e^x$$

$$-2A = 3$$

$$A = -\frac{3}{2} \qquad y_p = -\frac{3}{2}e^x$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{2}e^x}$$

notice if $f(x) = \sin x$ or $f(x) = \cos x$

$\rightarrow$ $y_p = A \sin x$ or $y_p = A \cos x$

$y_p' = A \cos x$ or $y_p = -A \sin x$

note the form changed ($\sin x$ changed to $\cos x$ or vice versa)

the method relies on the form NOT changing

$\boxed{\text{fix: even if there is only sine or cosine on the right, we ALWAYS include BOTH}}$

because if $y_p = A \cos x + B \sin x$ linear combo of both

$y_p' = -A \sin x + B \cos x$ still linear combo of both example $y'' - y' - 2y = \cos x$

Same left side as prev. example

so $y_c = c_1 e^{-x} + c_2 e^{2x}$

$f(x)$ is only $\cos x$ but include BOTH $\cos x$ and $\sin x$ in y_p

so, $y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

Sub into $y'' - y' - 2y = \cos x$

$$-A\cos x - B\sin x + A\sin x - B\cos x - 2A\cos x - 2B\sin x = \cos x$$

$$(-3A - B)\cos x + (-3B + A)\sin x = (1)\cos x + (0)\sin x$$

$$\left.\begin{aligned} -3A - B &= 1 \\ -3B + A &= 0 \end{aligned}\right\} \therefore A = -\frac{3}{10} \quad B = -\frac{1}{10}$$

So, $\boxed{y = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{10}\cos x - \frac{1}{10}\sin x}$

example $y'' - y' - 2y = xe^{3x}$

same $y_c = c_1 e^{-x} + c_2 e^{2x}$

$f(x) = xe^{3x}$

$$= \underbrace{(x+0)}_{\text{1st-deg polynomial}}\underbrace{e^{3x}}_{\text{exponential w/ pwr } 3x}$$

use the same form for y_p

$y_p = (Ax + B)e^{3x}$

$$\left.\begin{aligned} y_p &= Axe^{3x} + Be^{3x} \\ y_p' &= \cdots \\ y_p'' &= \cdots \end{aligned}\right\} \text{sub into } y'' - y' - 2y = xe^{3x}$$

$\vdots$

$A = \frac{1}{4}$ $B = -\frac{5}{16}$

__example__ $y'' - y = \cosh(x)$

$\cosh(x) = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$

$\sinh(x) = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$

$\frac{d}{dx}\cosh(x) = \sinh(x)$

$\frac{d}{dx}\sinh(x) = \cosh(x)$

two choices:

$$y'' - y = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$$

$$y_c = c_1 e^{x} + c_2 e^{-x}$$

right side : $y_p = Ae^{x} + Be^{-x}$ (match form)

but it's the same as y_c

so not linearly indp

fix: same as w/ repeated roots → multiply by x

so, the correct form of y_p is

$$\boxed{y_p = Axe^{x} + Bxe^{-x}}$$

find y_p', y_p'', sub into diff. eq.

⋮

$$y_p = \frac{1}{4}xe^{x} - \frac{1}{4}xe^{-x}$$

Second option: $y'' - y = \cosh(x)$

$$y_c = C_1 e^{x} + C_2 e^{-x}$$

$\hookrightarrow e^{-x} = \cosh(x) - \sinh(x)$

$\downarrow$

rewrite as $e^{x} = \cosh(x) + \sinh(x)$

⋮

$$= C_1 \cosh(x) + C_2 \sinh(x)$$

$y_p = A\cosh(x) + B\sinh(x)$ same as w/ $\cos x$ and $\sin x$

duplication of y_c

fix: give x's

$$y_p = Ax\cosh(x) + Bx\sinh(x)$$

sub into $y'' - y = \cosh(x)$

⋮

$$A = 0, \quad B = \frac{1}{2}$$

example $y^{(5)} + 2y^{(3)} + 2y'' = 3x^2 - 1$

form of particular solution

$$y_p = Ax^2 + Bx + C$$

check if this duplicates any part of y_c

$$y_c: \quad r^5 + 2r^3 + 2r^2 = 0$$

$$r^2(\underbrace{r^3 + 2r + 2}) = 0$$

$\underbrace{0, \; 0}$

3 nonzero roots
(no chance for y_p to duplicated these)

$$y_c = \underbrace{c_1 + c_2 x}_{\text{if } y_p \text{ copies this}} + c_3 \cdots + c_4 \cdots + c_5 \cdots$$

$y_p = Ax^2 + Bx + C$

duplicates c_1 in y_c

duplicates c_2x in y_c

if we only multiply C by x then Cx still duplicates c_2x y_c and duplicates Bx in y_p

→ multiply the entire y_p by addition x

$y_p = x(Ax^2 + Bx + C)$ did this solve the duplication prob w/ y_c?

yes, Cx still copies c_2x

fix: throw another x at it

$$\boxed{y_p = x^2(Ax^2 + Bx + C)}$$

now there is no more duplication

sub into $y^{(5)} + 2y^{(3)} + 2y'' = 3x^2 - 1$